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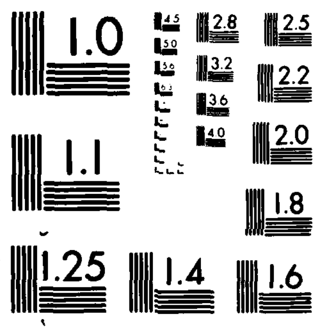

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APPLICABILITY OF THE FINITE ELEMENT CONCEPT TO HYPERBOLIC EQUATIONS

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The report analyses by means of examples the applicability of the finite element method (in the form of a weighted residual approach) to hyperbolic equations, using rectangular elements and bi-linear, bi-quadratic or bi-cubic shape functions. For sinusoidal initial conditions the errors are discussed for semi and fully discretized approximating equations. All methods have appreciable errors if the wave lengths are short. For semi-discretized methods, higher order elements give more accurate results at intermediate wave lengths. The fully discretized version for cubic elements becomes unstable, unless it is carried

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out as a combination of collocation and weighted residual methods. An example of a different kind shows the character of perturbations as one approaches the sonic line. A rationale for the choice of weight functions can be obtained by relating them to the Green's function. In two-dimensional problems, one can improve the cancellation of long distance effects of truncation errors by choosing characteristics as element boundaries.

PREFACE

The work was performed in 1979 under Grant AFOSR 78-3524 to the University of Dayton for the Applied Mathematics Group, Analysis and Optimization Branch, Structures and Dynamics Division, Flight Dynamics Laboratory under project 2304N1 and work unit 2304N110. Dr. Karl G. Guderley was Principal Investigator. The work of Donald S. Clemm of the Applied Mathematics Group was carried out at the Flight Dynamics Laboratory under work unit 2304N102, Mathematical Problems in Fluid Dynamics.

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SUMMARY

The report analyzes by means of examples the applicability of the finite element concept to hyperbolic equations. One of the aims is to provide an appreciation of the basic phenomena. The elements chosen are quadrangles with bilinear, biquadratic or bicubic shape functions. The equation discussed most is the two-dimensional wave equation. Variational approaches are rejected because they refer to problems that are not well posed; instead the method of weighted residuals with rather general weight functions is adopted. The equation of convergence which poses difficulty if an extremum formulae exists, has particular importance for hyperbolic problems because the exact solutions are not damped and truncation errors may cause a neutrally stable solution to become unstable. Besides stability the question of accuracy is explored. It determines, for a stable method, the admissible element size. Hyperbolic problems have equal waviness in the space and in the time direction. Therefore, one has, from the point of view of accuracy, a natural limitation of the Courant number to 1, even if this limitation is not needed for stability reasons. First a number of semi-discretized approaches (discretized in space but not in time) are investigated. The solutions always have dispersive character (no damping, but wave velocities different from the actual one). The ratio of the wave velocities obtained from the original partial differential by the numerical approach provides criterion for accuracy. The error in the wave velocity for long waves is much smaller for quadratic and cubic shape functions, even if one takes the fact into account that for quadratic and cubic shape functions the number of elements needed to give the same resolution in Fourier components is only one-half of that for linear wave functions. (The number of parameters describing the solutions at a given time is the same for the same resolution.) Short wave errors are nearly the same for different approaches. In a semidiscretized method one is led to a system of ordinary

differential equations (in implicit form). The solution must proceed in small steps because of the latent presence of particular solutions of the homogeneous system with a short wave length. Implicit procedures do not automatically insure the applicability of large time steps. Short waves (which in any case are inaccurate) can be suppressed if one works with third degree splines rather than cubic weight functions, which permit discontinuities in the second derivatives at the grid points. Next the time integration by finite elements is discussed. (It can be separated from the space discretization if one deals with bilinear, biquadratic or bicubic elements.) For linear elements the time dependence can be treated in the same manner as the space dependence. For Courant number 1, the discretization errors for time and space cancel each other. The method becomes unstable if the Courant number exceeds one. From a heuristic point of view such an approach is somewhat suspect, for it does not quite fit the idea of a marching procedure (which can always be carried out in hyperbolic problems). An analyogous approach for third degree elements is always unstable. A modified approach for linear elements which avoids this heuristic discrepancy gives stable but strongly damped particular solutions. From the point of accuracy, one would therefore need rather small elements. A similar approach for third degree shape functions converges for all Courant numbers and gives tolerable results up to Courant number one, if one takes constant weight throughout the new time interval in combination with a collocation condition for the new time point. An example discusses the choice of the mesh if , in a steady flow field, the Mach number approaches one at a certain line. Even then, one can always find a mesh which guarantees stability. In the example under discussion the particular solutions belonging to the highest frequencies (which are prone to cause instability) are of importance only in the immediate vicinity of the sonic line. A rational basis for the choice of weight functions can be

obtained by relating them to the Green's functions. It is possible to find weight functions which suppress the dominant long distance effect of a residual. For the Helmholtz equation this is only approximately correct and requires that the elements be sufficiently small. For the hyperbolic problem this cannot be done because the hyperbolic distance is zero for points which lie on the same characteristic. A better cancellation of long range effects in two-dimensional problems can be obtained if one chooses characteristics as element boundaries. The weight functions should then be constant in one of the characteristic directions. The idea cannot be carried over directly to the three-dimensional problem.

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SECTION I
INTRODUCTION

This report discusses the application of the finite element concept to hyperbolic problems. Some of the observations are probably familiar to those who have been working in this area; nevertheless the author hopes that the fairly systematic discussion carried out here has some value in so far as it clearly shows certain inherent difficulties.

An attempt to develop a general intuitive background for this problem area will be made in Section X. In the introduction we restrict ourselves to one general observation which motivates the choice of the approaches to be studied. In the classical applications of the finite element method to elliptic equations, the problem is governed by an extremum principle. This gives a guarantee of convergence, provided that the numerical process imitates the search for an extremum. (Of course the distance definition with respect to which convergence is obtained need not coincide with the error characterization desirable for an engineering application of the results.) For the hyperbolic problem no extremum formulation exists. In the classical examples one can define a functional which is stationary if one imposes suitable boundary conditions. But these boundary conditions are not identical with those of a well posed hyperbolic problem. For practical work a mere variational formulation has only limited usefulness to begin with (because of the absence of guaranteed convergence), but here the variational formulation does not even apply to the problem at hand. Reddy * has found a variational formulation for some hyperbolic equations in which the boundary conditions are properly taken into account. A simple example is shown in Appendix I, but

* (1) Reddy, J. N., A Note on Mixed Variational Principles for Initial-Value Problems, Quarterly Journal Mech. Appl. Math., Vol. XXVIII, Pt. 1, 1975, pp. 123-132.

because of the lack of an extremum property even this formulation does not seem useful from a numerical point of view.

Sections II through V study, for the wave equation, different approaches by semi-discretization; that is, the finite element concept is applied only to the spatial direction. This leads to systems of ordinary differential equations with the time as independent variable. It is assumed that this system is solved by an accurate integration method. In Section VI the finite element concept is applied also in the time direction. This discussion becomes fairly simple because of the special choice of bilinear, biquadratic or bicubic shape functions. The results show the importance of the Courant number. Sometimes the occurrence of large Courant numbers is unavoidable. Section VII gives details for a simple example of this kind. Section VIII tries to provide a rationale for the choice of the weight functions by relating these to the Green's function. In Section IX the finite element concept is combined with the idea of characteristics. The concluding Section X develops an intuitive picture of the problem area. In addition, it summarizes the specific results obtained.

A number of detailed studies are included in the form of appendices. We already mentioned that Appendix I gives an example for the variational formulation of Reddy. Appendix II, following an idea of Soliman, shows that rather general boundary conditions (except for Dirichlet conditions) fit smoothly into the concept of weighted residuals. Appendix III gives a number of results which can serve to check certain formulae which occur in the main body of the report. Appendix IV shows that the study of integration methods (including those based on the finite element concept) for systems of ordinary differential equations can be reduced to the discussion of a single scalar equation. Appendix V discusses Green's functions for the wave equation in two and three dimensions.

The present study therefore applies the finite element concept in the form of a method of weighted residuals. In a variational formulation the weight functions which are applied to the residuals are always identical with the shape functions used to represent the solutions (if not directly, then at least implicitly). In the absence of an extremum formulation this choice loses its usefulness. We shall admit weight functions of greater generality. (The term shape functions will be restricted to expressions which serve to represent the solutions, the word weight function is self-explanatory.) In classical applications the weight functions have finite support. They are $C^{(1)}$ or $C^{(2)}$ for partial differential equations of order two or four, respectively. This holds in particular at the edges of the region of support. This latter continuity requirement will no longer be imposed. In the integration by parts needed in order to treat derivatives of generalized functions correctly, certain additional terms outside of the integrals at the element boundaries are encountered. With the traditional choice of the weight functions (and of the shape functions), these terms will vanish. With the present more general choice of weight functions these terms must be taken into account. In the author's opinion, this is a technicality which should not preclude the use of such general weight functions.

By their very nature, hyperbolic equations lend themselves to a marching procedure. The solution at a certain time is not influenced by what happens at a later time. We have limited our discussion to approaches which can be interpreted in this sense. The procedures obtained in this manner need not be stable. A major part of the present work is the study of the stability (and of the accuracy) of different methods in a simple sample problem. In most of the report we deal with rectangular elements and bilinear, biquadratic or bicubic shape functions.

SECTION II
EQUATIONS OBTAINED BY SEMIDISCRETIZATION

In the examples of this report we shall study the wave equation

$$\phi_{yy} - \phi_{tt} = 0$$

(If one replaces ϕ_{tt} by ϕ_{xx} then one obtains the linearized equation of two dimensional supersonic flow at a free stream Mach number $\sqrt{2}$.) Practically in all cases one restricts stability discussions to equations with constant coefficients on the basis of the argument that in a restricted region and for a very fine mesh the coefficients are practically constant. The choice of such a simple equation is therefore not more restrictive then the usual stability discussions. We assume that the region extends in the y direction from $-\infty$ to $+\infty$. In the t direction the solution is to be determined for a finite interval. In this section an approach by semidiscretization is investigated; the differentiation with respect to the t direction is retained while one uses a finite element representation for the y dependence. Such an approach is frequently used for problems with two or three space dimensions.

In this section we shall write down the resulting systems of ordinary differential equations for different choices of the (solution) shape functions and of the weight functions, and also for a semidiscretized finite difference approach. The following problems will be treated.

1. The y derivative is replaced by a finite difference approximation.

2. ϕ is approximated by a piecewise linear function, ϕ is continuous

   a. the weight functions are identical with the shape functions

b. the weight functions are constant over intervals of a length of the gridsize which straddles the points where the first derivatives are discontinuous

3. ϕ is approximated by piecewise third order polynomials, ϕ and ϕ_y are continuous

a. the weight functions are identical with the shape functions

b. and c. the weight functions are constant over intervals of a length equal to half of the grid size. Cases b and c differ by the position of these intervals.

4. ϕ is approximated by piecewise second degree polynomials, ϕ is continuous.

a. the weight functions are related to the shape functions

b. the weight functions are constant over intervals of a length equal to half of the grid size.

5. ϕ is approximated by third degree polynomials, ϕ, ϕ_y, and ϕ_{yy} are continuous (spline approximation)

a. the weight functions are related to the shape functions

b. the weight functions are constant over an interval equal to the grid size.

6. ϕ is approximated by piecewise second degree polynomials, ϕ and ϕ_y are continuous

a. linear weight function

b. constant weight function

7. ϕ is approximated by third degree splines, and a collocation method is applied.

As usual the solutions are represented by a combination of shape functions, defined for the individual intervals (elemental shape functions). The formulae arising by this standard procedure are listed below for the convenience of a reader who wants to check the details. They are readily tested by the requirements that they give exact representations of the operators involved, for polynomials of a degree equal to that of the shape functions. This is done in Appendix III.

The interval in the y direction is h. The value of y at the k^{th} grid point is denoted by y_k. For the vicinity of the point y_k (usually for the interval $y_{k-1} \leq y \leq y_{k+1}$ we set

$$\Delta y = y - y_k \tag{2}$$

The elemental shape functions are written as functions of $(\Delta y/h)$. A prime denotes the derivative with respect to $(\Delta y/h)$. Furthermore, let

$$\begin{aligned} \phi_k(t) &= \phi(y_k, t) \\ \phi_k'(t) &= h\, \partial\phi(y_k, t)/\partial y \end{aligned} \tag{3}$$

The elemental shape functions are given by the following expressions.

Linear elemental shape functions (Figure 1)

$$\begin{aligned} N_1(\Delta y/h) &= 1 - \Delta y/h \\ N_2(\Delta y/h) &= \Delta y/h \end{aligned} \tag{4}$$

They satisfy

$$\begin{aligned} N_1(0) &= 1 \; ; \quad N_1(1) = 0 \\ N_2(0) &= 0 \; ; \quad N_2(1) = 1 \end{aligned}$$

Quadratic elemental shape functions (Figure 2)

$$\begin{aligned} N_1(\Delta y/h) &= 1 - 3(\Delta y/h) + 2(\Delta y/h)^2 \\ N_2(\Delta y/h) &= -(\Delta y/h) + 2(\Delta y/h)^2 \\ N_3(\Delta y/h) &= 4(\Delta y/h) - 4(\Delta y/h)^2 \end{aligned} \tag{5}$$

They satisfy

$$\begin{aligned} N_1(0) &= 1, \quad N_1(1/2) = 0, \quad N_1(1) = 0 \\ N_2(0) &= 0, \quad N_2(1/2) = 0, \quad N_2(1) = 1 \\ N_3(0) &= 0, \quad N_3(1/2) = 1, \quad N_3(1) = 0 \end{aligned}$$

Third degree elemental shape functions (Figure 3)

$$
\begin{aligned}
N_1(\Delta y/h) &= 1 \qquad\qquad\qquad\qquad - 3(\Delta y/h)^2 + 2(\Delta y/h)^3 \\
N_2(\Delta y/h) &= \qquad\qquad\qquad\qquad\quad 3(\Delta y/h)^2 - 2(\Delta y/h)^3 \\
N_3(\Delta y/h) &= \qquad\qquad (\Delta y/h) - 2(\Delta y/h)^2 + (\Delta y/h)^3 \\
N_4(\Delta y/h) &= \qquad\qquad\qquad\qquad\quad - (\Delta y/h)^2 + (\Delta y/h)^3
\end{aligned}
\tag{6}
$$

They satisfy

$$
\begin{aligned}
&N_1(0) = 1, \quad N_1'(0) = 0, \quad N_1(1) = 0, \quad N_1'(1) = 0 \\
&N_2(0) = 0, \quad N_2'(0) = 0, \quad N_2(1) = 1, \quad N_2'(1) = 0 \\
&N_3(0) = 0, \quad N_3'(0) = 1, \quad N_3(1) = 0, \quad N_3'(1) = 0 \\
&N_4(0) = 0, \quad N_4'(0) = 0, \quad N_4(1) = 0, \quad N_4'(1) = 1
\end{aligned}
$$

With these characterizations one readily obtains the following identities which can be used for checking purposes.

Linear elemental shape function

$$
\begin{aligned}
1 &= N_1 + N_2 \\
\Delta y/h &= N_2
\end{aligned}
$$

Quadratic elemental shape functions

$$
\begin{aligned}
1 &= N_1 + N_2 + N_3 \\
\Delta y/h &= N_2 + N_3/2 \\
(\Delta y/h)^2 &= N_2 + N_3/4
\end{aligned}
$$

Third degree shape functions

$$
\begin{aligned}
1 &= N_1 + N_2 \\
\Delta y/h &= N_2 + N_3 + N_4 \\
(\Delta y/h)^2 &= N_2 + 2N_4 \\
(\Delta y/h)^3 &= N_2 + 3N_4
\end{aligned}
$$

The solutions are characterized:
in the linear case by the values of $\phi_k(t)$,
in the quadratic case by the values of $\phi_k(t)$ and $\phi_{k+1/2}(t)$,
in the third degree case by the values of $\phi_k(t)$ and $\phi'_k(t)$
The values of k range in our examples from $-\infty$ to $+\infty$.
Notice that one has two parameters per interval for the quadratic as well as the third degree case.

Let $w(\Delta y/h)$ be one of the weight functions. Then one obtains an approximating equation

$$\int_{-\infty}^{+\infty} \frac{\partial^2\phi}{\partial t^2} w(\Delta y/h)\, d(\Delta y/h) - \int_{-\infty}^{+\infty} \frac{\partial^2\phi}{\partial y^2} w(\Delta y/h)\, d(\Delta y/h) = 0 \qquad (7)$$

For each choice of k and each weight function we obtain one condition. Here one must substitute for ϕ the approximations listed above. The weight functions w have finite support (that is they are different from zero only over a finite interval) and the integrations are only needed for finite intervals. For linear and quadratic shape functions the term ϕ_{yy} will lead to delta functions at the grid points, the integrals must then be evaluated in the sense of generalized functions.

Case 1. Finite difference approximation.
One obtains

$$\frac{\partial^2\phi_k}{\partial t^2} - h^{-2}(\phi_{k-1} - 2\phi_k + \phi_{k+1}) = 0 \quad ; \; -\infty < k < \infty \qquad (8)$$

Case 2. Linear shape functions
The shape function belonging to the point y_k is given by Figure 4.

$$N_1(\Delta y/h) + N_2((\Delta y/h)+1)$$

(It is understood that the elemental shape functions N_i are zero, unless their argument lies between zero and one.)
Accordingly, N_1 ($\Delta y/h$) is different from zero only for

$y_k < y < y_{k+1}$, N_2 is different from zero only for $y_{k-1} < y < y_k$. One has in the interval $y_{k-1} \leq y \leq y_{k+1}$

$$\phi(y,t) = \phi_{k-1}(t)\, N_1((\Delta y/h)+1) + \phi_k(t)\,[N_2((\Delta y/h)+1) + N_1(\Delta y/h)] + \phi_{k+1}(t)\, N_2(\Delta y/h) \tag{9}$$

<u>Case 2a</u>

The weight function belonging to point y_k is given by (Figure 4)

$$w(\Delta y/h) = N_1(\Delta y/h) + N_2((\Delta y/h)+1)$$

or

$$\begin{aligned} w(\Delta y/h) &= 1 + (\Delta y/h) \quad ; \quad y_{k-1} < y < y_k \\ w(\Delta y/h) &= 1 - (\Delta y/h) \quad ; \quad y_k < y < y_{k+1} \end{aligned} \tag{10}$$

Otherwise they are zero.

One obtains

$$\frac{d^2}{dt^2}\left\{\phi_k + \frac{1}{6}(\phi_{k-1} - 2\phi_k + \phi_{k+1})\right\} - h^{-2}\left\{\phi_{k-1} - 2\phi_k + \phi_{k+1}\right\} = 0 \tag{11}$$

For a test see Appendix III.

<u>Case 2b</u>

The weight function belonging to point y_k is given by (Figure 5)

$$w(\Delta y/h) = 1 \quad ; \quad -\tfrac{1}{2} < (\Delta y/h) < \tfrac{1}{2} \tag{12}$$

One obtains

$$\frac{d^2}{dt^2}\left\{\phi_k + \frac{1}{8}(\phi_{k-1} - 2\phi_k + \phi_{k+1})\right\} - h^{-2}\left\{\phi_{k-1} - 2\phi_k + \phi_{k+1}\right\} = 0 \tag{13}$$

For a test see Appendix III.

Case 3 Third degree elements

The shape functions belonging to point y_k are (Figure 6)

$$N_1(\Delta y/h) + N_2((\Delta y/h)+1)$$

and

$$N_3(\Delta y/h) + N_4((\Delta y/h)+1$$

In the interval $y_{k-1} < y < y_{k+1}$ one has

$$\begin{aligned}\phi(y,t) = {} & \phi_{k-1}(t) N_1((\Delta y/h)+1) + \phi'_{k-1}(t) N_3((\Delta y/h)+1) \\ & + \phi_k(t)[N_2((\Delta y/h+1) + N_1(\Delta y/h)] + \phi'_k(t)[N_4((\Delta y/h+1) + N_3(\Delta y/h)] \\ & + \phi_{k+1}(t) N_2(\Delta y/h) + \phi'_{k+1}(t) N_4(\Delta y/h) \end{aligned} \qquad (14)$$

Case 3a

The weight functions belonging to point y_k are (Figure 6)

$$w_1(\Delta y/h) = N_1(\Delta y/h) + N_2((\Delta y/h)+1) \qquad (15a)$$

and

$$w_2(\Delta y/h) = N_3(\Delta y/h) + N_4((\Delta y/h)+1) \qquad (15b)$$

One obtains the following differential equations (two for every value of k)

$$\begin{aligned} & \frac{d^2}{dt^2}\left\{\phi_k + \frac{9}{70}(\phi_{k-1} - 2\phi_k + \phi_{k+1}) - \frac{13}{420}(\phi'_{k+1} - \phi'_{k-1})\right\} \\ & + h^{-2}\left\{-\frac{6}{5}(\phi_{k-1} - 2\phi_k + \phi_{k+1}) + \frac{1}{10}(\phi'_{k+1} - \phi'_{k-1})\right\} = 0 \quad ; -\infty < k < \infty \end{aligned} \qquad (16a)$$

$$\begin{aligned} & \frac{d^2}{dt^2}\left\{\frac{13}{420}(\phi_{k+1} - \phi_{k-1}) + \left[\frac{1}{210}\phi'_k - \frac{1}{140}(\phi'_{k-1} - 2\phi'_k + \phi'_{k+1})\right]\right\} \\ & + h^{-2}\left\{-\frac{1}{10}(\phi_{k+1} - \phi_{k-1}) + \left[\frac{1}{5}\phi'_k - \frac{1}{30}(\phi'_{k-1} - 2\phi'_k + \phi'_{k+1})\right]\right\} = 0 \quad ; -\infty < k < \infty \end{aligned} \qquad (16b)$$

Case 3b

The weight functions are given by (Figure 7)

$$w_1(\Delta y/h) = 1 \quad ; \quad -\tfrac{1}{2} < \Delta y/h < +\tfrac{1}{2} \tag{17a}$$

$$w_2(\Delta y/h) = \begin{cases} -1 & ; \quad -\tfrac{1}{2} < \Delta y/h < 0 \\ +1 & ; \quad 0 < \Delta y/h < \tfrac{1}{2} \end{cases} \tag{17b}$$

otherwise they are zero. These weight functions are obviously equivalent to

$w = 1; \; 0 < \Delta y/h < \tfrac{1}{2}$ and $w = 1; \; -\tfrac{1}{2} < \Delta y/h$

One obtains

$$\frac{d^2}{dt^2}\left\{\phi_k + \frac{3}{32}(\phi_{k-1} - 2\phi_k + \phi_{k+1}) - \frac{5}{192}(\phi'_{k+1} - \phi'_{k-1})\right\} \\ + h^{-2}\left\{-\frac{3}{2}(\phi_{k-1} - 2\phi_k + \phi_{k+1}) + \frac{1}{4}(\phi'_{k+1} - \phi'_{k-1})\right\} = 0 \quad ; -\infty < k < \infty \tag{18a}$$

$$\frac{d^2}{dt^2}\left\{\frac{3}{32}(\phi_{k+1} - \phi_{k-1}) + \left[\frac{1}{16}\phi'_k - \frac{5}{192}(\phi'_{k-1} - 2\phi'_k + \phi'_{k+1})\right]\right\} \\ + h^{-2}\left\{-\frac{3}{2}(\phi_{k+1} - \phi_{k-1}) + \left[3\phi'_k + \frac{1}{4}(\phi'_{k-1} - 2\phi'_k + \phi'_{k+1})\right]\right\} = 0 \quad ; -\infty < k < \infty \tag{18b}$$

Case 3c

The weight functions are given by (Figure 8)

$$w_1(\Delta y/h) = 1 \quad ; \quad -1/4 < \Delta y/h < 1/4 \\ w_2(\Delta y/h) = 1 \quad ; \quad 1/4 < \Delta y/h < 3/4 \tag{19}$$

otherwise they are zero.

One obtains

$$\frac{d^2}{dt^2}\left\{\frac{1}{2}\phi_k + \frac{7}{512}(\phi_{k-1} - 2\phi_k + \phi_{k+1}) - \frac{13}{3072}(\phi'_{k+1} - \phi'_{k-1})\right\} \\ + h^{-2}\left\{-\frac{9}{8}(\phi_{k+1} - 2\phi_k + \phi_{k-1}) + \frac{5}{16}(\phi'_{k+1} - \phi'_{k-1})\right\} = 0 \quad ; -\infty < k < \infty \tag{20a}$$

and

$$\frac{d^2}{dt^2}\left\{\frac{1}{4}(\phi_k+\phi_{k+1})-\frac{11}{192}(\phi'_{k+1}-\phi'_{k-1})\right\}-h^{-2}\left\{\frac{1}{2}(\phi'_{k+1}-\phi'_{k-1})\right\}=0 \tag{20b}$$

$$-\infty < k < \infty$$

<u>Case 4</u>. Second degree elements

In the interval $y_{k-1} < y < y_{k+1}$ one has the following representation for ϕ

$$\phi(y,t)=\phi_{k-1}(t)N_1((\Delta y/h)+1)+\phi_k(t)[N_1(\Delta y/h)+N_2((\Delta y/h)+1)]+\phi_{k+1}(t)N_2(\Delta y/h)$$
$$+\phi_{k-\frac{1}{2}}(t)N_3(\Delta y/h+1)+\phi_{k+\frac{1}{2}}(t)N_3(\Delta y/h)$$

<u>Case 4a</u>

The weight functions are given by (Figure 9)

$$w_1(\Delta y/h)=\begin{cases}1+(\Delta y/h) & ;\ -1<\Delta y/h<0\\ 1-(\Delta y/h) & ;\ 0<\Delta y/h<1\end{cases} \tag{21a}$$

$$w_2(\Delta y/h)=N_3(\Delta y/h)=4(\Delta y/h)-4(\Delta y/h)^2 \tag{21b}$$

Notice that the weight function w_1 is a linear combination of elemental quadratic shape functions. This choice has been made to avoid negative weights.

One obtains

$$\frac{d^2}{dt^2}\left\{\frac{1}{3}\phi_k+\frac{1}{3}(\phi_{k-\frac{1}{2}}+\phi_{k+\frac{1}{2}})\right\}-h^{-2}\left\{\phi_{k-1}-2\phi_k+\phi_{k+1}\right\}=0 \tag{22a}$$

$$-\infty < k < \infty$$

and

$$\frac{d^2}{dt^2}\left\{\frac{1}{15}(\phi_k+\phi_{k+1})+\frac{8}{15}\phi_{k+\frac{1}{2}}\right\}+h^{-2}\left\{-\frac{8}{3}(\phi_k+\phi_{k+1})+\frac{16}{3}\phi_{k+\frac{1}{2}}\right\}=0 \tag{22b}$$

$$-\infty < k < \infty$$

Case 4b

The weight functions are given by (Figure 8)

$$w_1 = 1 \, , \quad -\tfrac{1}{4} < \Delta y/h < +\tfrac{1}{4} \tag{23a}$$

$$w_2 = 1 \, , \quad \tfrac{1}{4} < \Delta y/h < \tfrac{3}{4} \tag{23b}$$

One obtains

$$\frac{d^2}{dt^2}\left\{\frac{7}{24}\phi_k - \frac{1}{48}(\phi_{k-1} - 2\phi_k + \phi_{k+1}) + \frac{5}{48}(\phi_{k+\frac{1}{2}} + \phi_{k-\frac{1}{2}})\right\} \tag{24a}$$
$$+ h^{-2}\left\{4\phi_k - 2(\phi_{k+\frac{1}{2}} + \phi_{k-\frac{1}{2}})\right\} = 0 \quad , -\infty < k < \infty$$

$$\frac{d^2}{dt^2}\left\{\frac{1}{48}(\phi_k + \phi_{k+1}) + \frac{11}{24}\phi_{k+\frac{1}{2}}\right\} + h^{-2}\left\{-2(\phi_k + \phi_{k+1}) + 4\phi_{k+\frac{1}{2}}\right\} = 0 \tag{24b}$$
$$-\infty < k < \infty$$

Case 5. Third degree shape functions with continuous second derivatives

The jump of the second derivative with respect to $\Delta y/h$ at a point $y = y_k$ is given by the left side of the following expression

$$6(\phi_{k+1} - \phi_{k-1}) - 8\phi_k' - 2(\phi_{k+1}' + \phi_{k-1}') = 0 \tag{25}$$

It is not difficult to devise a test for the correctness of this expression similar to the tests derived in Appendix III. This expression is now combined with Eqs. (16a) pertaining to the weight function eq. (15a) and with Eq. (18a) pertaining to weight functions Eq. (17a). Now one has the systems of equations.

Case 5a

$$\frac{d^2}{dt^2}\left\{\phi_k + \frac{9}{70}(\phi_{k-1} - 2\phi_k + \phi_{k+1}) - \frac{13}{420}(\phi_{k+1}' - \phi_{k-1}')\right\}$$
$$+ h^{-2}\left\{-\frac{6}{5}(\phi_{k-1} - 2\phi_k + \phi_{k+1}) + \frac{1}{10}(\phi_{k+1}' - \phi_{k-1}')\right\} = 0 \tag{26a}$$

and

$$6(\phi_{k+1}-\phi_{k-1})-8\phi_k'-2(\phi_{k+1}'+\phi_{k-1}')=0 \qquad (26b)$$

$$-\infty < k < \infty$$

Case 5b

$$\frac{d^2}{dt^2}\left\{\phi_k+\frac{3}{32}(\phi_{k-1}-2\phi_k+\phi_{k+1})-\frac{5}{192}(\phi_{k+1}'-\phi_{k-1}')\right\} \qquad (27a)$$
$$+h^{-2}\left\{-\frac{3}{2}(\phi_{k-1}-2\phi_k+\phi_{k+1})+\frac{1}{4}(\phi_{k+1}'-\phi_{k-1}')\right\}$$

$$6(\phi_{k+1}-\phi_{k-1})-8\phi_k'-2(\phi_{k+1}'+\phi_{k-1}')=0 \qquad (27b)$$
$$-\infty < k < \infty$$

The weight functions of the case 3c are unsuited to such an approach.

In practical applications there are always a finite number of values ϕ_k and ϕ_k'. Then one can express the values of ϕ_k' in terms of the values of ϕ_ℓ by means of the second equation. One thus obtains a system of equations for the ϕ_k's only. For the present discussions this is not practical.

Case 6.

Quadratic shape functions studied in case 4 allow the first derivatives to be discontinous. It might be a simplification if one imposes the condition of continuity of the first derivatives. The jump of the first derivative at the point $y = y_k$ is given by the left hand side of the following equation

$$-6\phi_k-(\phi_{k+1}+\phi_{k-1})+4(\phi_{k+\frac{1}{2}}+\phi_{k-\frac{1}{2}})=0 \qquad (28)$$

Case 6a. (linear weight functions)

We use this condition in combination with Eq. (22a) which holds for the weight functions Eq. (21a), Figure 9a.

One then obtains

$$\frac{d^2}{dt^2}\left\{\frac{1}{3}\phi_k + \frac{1}{3}\left(\phi_{k+\frac{1}{2}} + \phi_{k-\frac{1}{2}}\right)\right\} - h^{-2}\left\{\phi_{k-1} - 2\phi_k + \phi_{k+1}\right\} = 0 \qquad (29a)$$

$$-6\phi_k - (\phi_{k-1} + \phi_{k+1}) + 4\left(\phi_{k+\frac{1}{2}} + \phi_{k-\frac{1}{2}}\right) = 0 \qquad (29b)$$

$$-\infty < k < \infty$$

<u>Case 6b</u>. Constant weight function

The expression Eq. (23a) is not suitable as sole weight function because it fails to cover the whole interval. We use, instead, the weight Eq. (17a) (Fig. 7a) and obtain

$$\frac{d^2}{dt^2}\left\{\frac{5}{12}\phi_k - \frac{1}{24}(\phi_{k+1} + \phi_{k-1}) + \frac{1}{3}\left(\phi_{k+\frac{1}{2}} + \phi_{k-\frac{1}{2}}\right)\right\} \qquad (30a)$$
$$- h^{-2}\left\{\phi_{k+1} - 2\phi_k + \phi_{k-1}\right\} = 0$$

$$-6\phi_k - (\phi_{k+1} + \phi_{k-1}) + 4\left(\phi_{k+\frac{1}{2}} + \phi_{k-\frac{1}{2}}\right) = 0 \qquad (30b)$$

<u>Case 7</u>.

The second derivative is defined everywhere if one deals with third degree polynomials and continuous first and second derivatives. It is then possible to use such an approximation in order to define the second derivatives at the grid points and to apply a collocation method. For ϕ_{yy} at point y_k one obtains from the interval $y_k < y < y_{k+1}$

$$\phi_{yy}\Big|_{y_k+\varepsilon} = h^{-2}\left(-6\phi_k + 6\phi_{k+1} - 4\phi_k' - 2\phi_{k+1}'\right)$$

correspondingly

$$\phi_{yy}\Big|_{y_k-\varepsilon} = h^{-2}\left(-6\phi_k + 6\phi_{k-1} + 4\phi_k' + 2\phi_{k-1}'\right)$$

Eq. (28) guarantees that these expressions be the same. For the sake of symmetry we write $\psi_{yy}(y_k)$ as the average of these values. Then one obtains from Eq. (1) (with a change of sign)

$$\frac{d^2\phi_k}{dt^2} - h^{-2}\{3(\phi_{k-1} - 2\phi_k + \phi_{k+1}) - (\phi'_{k+1} - \phi'_{k-1})\} = 0 \tag{31a}$$

together with

$$6(\phi_{k+1} - \phi_{k-1}) - 8\phi'_k - 2(\phi'_{k+1} + \phi'_{k-1}) = 0$$
$$-\infty < k < \infty \tag{31b}$$

SECTION III

STABILITY TESTS FOR THE SEMIDISCRETIZED PROBLEM

A stability analysis can be carried out in the usual manner. The interval in the y direction is assumed to go from $-\infty$ to $+\infty$. Alternatively, one might assume periodicity conditions. Then one sets

$$\phi_k(t) = C \exp(i\mu kh) \exp(i\nu t) \tag{32a}$$

and if needed

$$\phi_k'(t) = i\tilde{C} \exp(i\mu kh) \exp(i\nu t) \tag{32b}$$

and computes the values of ν in dependence upon μ. The method is stable, if ν is real or has a positive imaginary part. If μ is small (long waves), then one expects that $\nu = \pm \mu$. In a precise solution of the original problem this result should be found for all values of μ. In an approximate method this relation will be satisfied only approximately. This gives an insight into the errors introduced by different approaches.

One obtains with Eqs. (32)

$$\phi_{k-1} - 2\phi_k + \phi_{k+1} = C \exp(i\mu kh) \exp(i\nu t)(-4) \sin^2(\mu h/2) \tag{33}$$

$$\phi_{k+1} - \phi_{k-1} = C \exp(i\mu kh) \exp(i\nu t)(2i \sin \mu h) \tag{34}$$

$$\phi_{k-1}' - 2\phi_k' + \phi_{k+1}' = \tilde{C} \exp(i\mu kh) \exp(i\nu t)(-4i) \sin^2(\mu h/2) \tag{35}$$

$$\phi_{k+1}' - \phi_{k-1}' = \tilde{C} \exp(i\mu kh) \exp(i\nu t)(-2) \sin \mu h \tag{36}$$

One obtains:

Case 1. (finite differences, Eq. (8))

$$\nu^2 h^2 = 4\sin^2(\mu h/2) \tag{36}$$

Case 2a. (linear shape functions, linear weight functions Eq. (11))

$$\nu^2 h^2 = \frac{4\sin^2(\mu h/2)}{1-(2/3)\sin^2(\mu h/2)} \tag{37}$$

Case 2b. (linear shape functions, constant shape functions, Eq. (13))

$$\nu^2 h^2 = \frac{4\sin^2(\mu h/2)}{1-(1/2)\sin^2(\mu h/2)} \tag{38}$$

Case 3a. (third degree shape functions, weight functions derived from shape functions Eqs (16))

$$\left\{ \begin{bmatrix} (24/5)\sin^2(\mu h/2) & -(1/5)\sin\mu h \\ (1/5)\sin(\mu h) & (1/5)+(2/15)\sin^2(\mu h/2) \end{bmatrix} - \nu^2 h^2 \begin{bmatrix} 1-(18/35)\sin^2(\mu h/2) & (13/210)\sin(\mu h) \\ (13/210)\sin(\mu h) & (1/210)+(1/35)\sin^2(\mu h/2) \end{bmatrix} \right\} \begin{Bmatrix} C \\ \tilde{C} \end{Bmatrix} = 0 \tag{39}$$

In order for this system of equations to have a nontrivial solution for the vector $[C, \tilde{C}]^+$, its determinant must vanish. This gives a quadratic equation for $\nu^2 h^2$. For the numerical evaluation this determinant is a suitable starting point.

Case 3b. (third degree shape functions, constant weight functions Eqs. (18))

$$\left| \begin{bmatrix} 6\sin^2(\mu h/2) & -(1/2)\sin\mu h \\ -3\sin(\mu h) & 3-\sin^2(\mu h/2) \end{bmatrix} - \nu^2 h^2 \begin{bmatrix} 1-(3/8)\sin^2(\mu h/2) & (5/96)\sin\mu h \\ (3/16)\sin(\mu h) & (1/16)+(5/48)\sin^2(\mu h/2) \end{bmatrix} \right| = 0 \tag{40}$$

Case 3c. (third degree shape functions, constant weight functions Eqs. (20))

$$\left| \begin{bmatrix} (9/2)\sin^2(\mu h/2) & -(5/8)\sin(\mu h) \\ 0 & \sin(\mu h/2) \end{bmatrix} - \nu^2 h^2 \begin{bmatrix} (1/2)-(7/128)\sin^2(\mu h/2) & (13/1536)\sin(\mu h) \\ (1/2)\cos(\mu h/2) & (11/196)\sin(\mu h/2) \end{bmatrix} \right| = 0 \tag{41}$$

Case 4. Quadratic shape functions

We set

$$\begin{aligned} \phi_k &= C\exp(i\mu kh)\exp(i\nu t) \\ \phi_{k+\frac{1}{2}} &= \tilde{C}\exp(i\mu(k+\tfrac{1}{2})h)\exp(i\nu t) \end{aligned} \tag{42}$$

Case 4a. (weight functions derived from shape functions, Eqs. (21))

$$\left| \begin{bmatrix} 4\sin^2(\mu h/2) & 0 \\ -(16/3)\cos(\mu h/2) & 16/3 \end{bmatrix} - \nu^2 h^2 \begin{bmatrix} 1/3 & (2/3)\cos(\mu h/2) \\ (2/15)\cos(\mu h/2) & 8/15 \end{bmatrix} \right| = 0 \tag{43}$$

Case 4b. (constant weight functions, Eqs. (23))

$$\left| \begin{bmatrix} 4 & -4\cos(\mu h/2) \\ -4\cos(\mu h/2) & 4 \end{bmatrix} - \nu^2 h^2 \begin{bmatrix} (7/24) + (1/12)\sin^2(\mu h/2) & (5/24)\cos(\mu h/2) \\ 1/24\cos(\mu h/2) & 11/24 \end{bmatrix} \right| = 0 \tag{44}$$

Case 5. (third degree shape functions with continuous second derivatives)

Eq. (25) which introduces continuity of the second derivatives, combined with Eqs. (32) gives

$$\tau = \frac{\sin(\mu h)}{1 - (2/3)\sin^2(\mu h/2)} \tag{45}$$

Case 5a. (weight function related to shape function)

One obtains from the first of Eqs. (39)

$$\nu^2 h^2 = \frac{(24/5)\sin^2(\mu h/2)\left(1 - (2/3)\sin^2(\mu h/2)\right) - (1/5)\sin^2(\mu h)}{\left(1 - (18/35)\sin^2(\mu h/2)\right)\left(1 - (2/3)\sin^2(\mu h/2)\right) + (13/210)\sin^2(\mu h)}$$

This can be simplified to

$$\nu^2 h^2 = \frac{4\sin^2(\mu h/2) - (12/5)\sin^4(\mu h/2)}{\left(1 - (98/105)\sin^2(\mu h/2)\right) + (10/105)\sin^4(\mu h/2)} \tag{46}$$

Case 5b. (weight function constant)

One obtains from the first row in the determinant Eq. (40)

$$\nu^2 h^2 = \frac{6\sin^2(\mu h/2)(1-(2/3)\sin^2(\mu h/2)) - (1/2)\sin^2(\mu h)}{(1-(3/8)\sin^2(\mu h/2))(1-(2/3)\sin^2(\mu h/2)) + (5/96)\sin^2(\mu h)}$$

This simplifies to

$$\nu^2 h^2 = \frac{4\sin^2(\mu h/2) - 2\sin^4(\mu h/2)}{1-(5/6)\sin^2(\mu h/2) + (1/4)\sin^4(\mu h/2)} \tag{47}$$

Case 6. (quadratic shape functions with continuous first derivatives

The condition(28) for continuity of the first derivatives in combination with Eq. (42) gives

$$\tilde{C} = \frac{1-(1/2)\sin^2(\mu h/2)}{\cos(\mu h/2)}\, C$$

Case 6a. (linear weight function)

One obtains from the first row in the determinant Eq. (43)

$$\nu^2 h^2 = \frac{4\sin^2(\mu h/2)}{1-(1/3)\sin^2(\mu h/2)} \tag{48}$$

Case 6b. (constant weight function)

The result is derived from Eq. (30a)

$$\nu^2 h^2 = \frac{4\sin^2(\mu h/2)}{1-(1/6)\sin^2(\mu h/2)} \tag{49}$$

Case 7. (third degree splines combined with collocation)

Here the condition (45) (continuity of second derivatives) applies again. One obtains from Eq. (31a)

$$\nu^2 h^2 = \frac{12\sin^2(\mu h/2)(1-(2/3)\sin^2(\mu h/2)) + 2\sin^2(\mu h)}{1-(2/3)\sin^2(\mu h/2)} = \frac{4\sin^2(\mu h/2)}{1-(2/3)\sin^2(\mu h/2)} \tag{50}$$

SECTION IV

DISCUSSION OF THE RELATIONS DERIVED IN SECTION III

The relations between νh and μh derived in the preceding section have been evaluated numerically. The information so obtained is valid only if one solves the systems of ordinary differential equations which are written down in Section II precisely. In reality, one must apply some approximation also for this purpose. Some observations regarding such integrations are made in the next section. The formulae in Section III are based on the hypothesis (32a) (supplemented by other expressions of a similar form) that is

$$\phi_k(t) = C \exp(i\mu kh) \exp(\pm i\nu t)$$

which corresponds to an approximate particular solution

$$\phi(y,t) = C \exp(i\mu y) \exp(\pm i\nu t) = C \exp(i\mu(y \pm (\nu/\mu)t)$$

The wave speed in this approximation is then $\pm\ \nu/\mu$, the wave speed for exact solutions of the original differential equation is $\pm$ 1.

In the approximations discussed so far, the wave speeds are real for all values of μ; there is no damping. (One would speak of a dispersive method because waves with different wave lengths do not remain together.) Because of the error in the wave speeds, one obtains a phase error in the wave after some time has elapsed, which is given by

$$\mu(1 - (\nu/\mu))t$$

In the graphs the value of ν/μ is shown, the phase error decreases more rapidly with μ than $\nu/\mu - 1$, because of the factor μ which occurs in the last expression. The contribution of a certain wave length to the solution becomes meaningless if the phase error exceeds some fairly small number (perhaps $\pi/6$ or smaller). One must make the grid size h small

enough so that for the time for which the solution is required, the phase error for the significant wave lengths stays small.

For the cases 1 and 2, one obtains only one value of $(\nu h)^2$ for each choice of (μh). For cases 3 and 4, one obtains a quadratic equation for $(\nu h)^2$. This requires an explanation.

Consider expressions which assume, at the grid points, the following values:

$$\phi_k = C\cos(\mu k h)$$
or
$$\phi_k = C\sin(\mu k h) \tag{51}$$

These expressions can be rewritten in the form

$$\phi_k = C\cos((\mu h + 2\pi n)k) = C\cos((-\mu h + 2\pi n)k)$$
$$\phi_k = C\sin((\mu h + 2\pi n)k) = -C\sin((-\mu h + 2\pi n)k) \tag{52}$$

Except for a trivial change of sign (in the sine function), one obtains the same values at the grid points y = kh, if one replaces μ by $-\mu$ and or changes μh by a multiple of 2π. The above expressions can originate from waves of the form

$$\phi = C\cos(\mu_1 y)$$
$$\phi = \pm C\sin(\mu_1 y) \tag{53}$$

where $\mu_1 = \mu \bmod \frac{2\pi}{h}$, or $\mu_1 = -\mu \bmod \frac{2\pi}{h}$.

This is the well known phenomenon of aliasing. For the evaluation of the formulae of the previous section, it is therefore sufficient if $0 \leq \mu \leq \pi$.

In general one will, of course, associate the values of νh obtained from these formulae with the smallest possible values of μ which are admissible according to the above formulae. But in principle, there is no reason to disregard other values of μ_1 right from the start. Of course one will consider only values of μ for which the pertinent values of ν give tolerable approximations to the wave speed.

In evaluating νh for quadratic and third order shape functions, one obtains two values of $\nu^2 h^2$ for one value of μh. One will surmise that one of these values νh belongs to μh, while the second one belongs to $2\pi-\mu h$. This is borne out by the graphs. For an analytical explanation consider the case $\mu = 0$. Then one has for $\mu_1 = \mu = 0$

$$\phi = \cos(\mu_1 y) = 1$$

hence

$$\phi_k = 1$$

$$\phi_k' = 0$$

and for $\mu_1 = (2\pi/h) - \mu \quad = (2\pi/h)$

$$\phi = \sin((2\pi/h)y$$

$$\phi_y = \frac{2\pi}{h} \cos((2\pi/h)y)$$

hence

$$\phi_k = \sin((2\pi/h)kh) = 0$$

$$\phi_k' = 2\pi \cos((2\pi/h)kh) = 1$$

The first expression $\phi_k = 1$ is exactly represented by $\phi = 1$ which is a third degree polynomial. The second expression gives within one interval the expression

$$\phi = 2\pi(N_3 + N_4) = 2\pi(\Delta y/h)(1-(\Delta y/h))(1-2(\Delta y/h))$$

It is shown in Figure 10, together with the function $\sin(2\pi y)$.

This result is not surprising. For second and third order degree shape functions, one has twice as many parameters per grid point than for first degree shape functions. Accordingly, one can approximate functions with a wave length one half of that which can be approximated by first degree shape functions.

The amount of numerical work to solve a system of equations with double the number of unknowns is, of course, at least twice as large. To compensate for this fact, one will take the interval in the y direction for second and third order shape functions twice as large as for linear shape functions. The scale of μh for second degree shape functions has therefore been chosen 1/2 of the μh scale for first degree shape functions.

Figure 11 gives the results for ν/μ versus μh for the cases 1 and 2. Curve a pertains to a finite difference semidiscretization, curve b belongs to case 2a (linear shape functions, linear weight functions), curve c to case 2b (linear weight functions, constant shape functions). Ideally, one should obtain 1 for all values of μh. The finite element approximations are somewhat better than the finite difference approximation, the region of values μh for which the ν/μ is nearly correct is rather limited. Incidentally, curve b is also obtained in case 7 (third order shape functions with a collocation method).

Figures 12 and 13 give results for third degree shape functions. Notice that in these cases μh ranges from zero to 2π and that at $\mu h = \pi$ one has a break. The reason is explained above. Curve 12a and 13a give the result for case 3a, weight functions derived from shape functions, curve 12b shows the result for case 3b, curve 13b gives the result for case 3c, both have constant weights but they are shifted with respect to the grid. In this case, weight functions

derived from shape functions are superior. It is rather disconcerting that a small change of the weight function as it occurs between Figures 12b and 13b, gives a rather significant difference in the results.

It is conceivable that under slightly changed circumstances the results would be different. Under the present circumstances the procedure of case 3a gives the best results, but the author is not sure whether or not the same behavior can be expected for more complicated problems. We explained above, why the region $0 < \mu h \leq 2\pi$ for the third degree shape functions is considered as equivalent to the region $0 < \mu h < \pi$ for linear functions. For shorter wave lengths the error in the wave speed is about the same for linear and third order shape functions. However, the values of μh for which long waves are adequately represented, that is where ν/μ is close to 1 is considerably enlarged for third degree elements.

Summarizing, one can say by taking third degree shape functions with twice the grid size instead of first order shape functions, one is able to represent in both cases the same waviness with respect to the y direction. The wave speed for short waves is falsified by about the same amount in either case. Long waves are less falsified for third degree elements than for first degree elements.

If one uses such a procedure, one must always take the mesh fine enough, so that the behavior of all the waves that are important for the solution is adequately represented. Third degree order elements then make it possible to use a far larger mesh (ultimately to work with fewer unknowns) than first degree elements. Because of rounding errors, one must expect shorter wave lengths to appear in the solutions, the propagation of short waves is falsified by the procedure. Such errors do not die out (as they would in an elliptic problem) but

at least their amplitude does not increase. Moreover, if such waves are due to rounding errors, then they are likely to have a random character so that they cancel in the average. The discretization error is already taken into account in the falsification of the wave speed.

Since short waves are wrongly represented for third degree shape functions, it might be desirable to suppress them altogether. This can be done by replacing one set of conditions which arise by the use of a certain weight function by the requirement that the second derivative be continuous. In this manner one connects the ϕ_k' 's with the ϕ_k's. In other words, one uses a third order spline representation. The results are shown in Figures 14 and 15. The requirement of continuity of the second derivative is a restriction of the space of functions that is available for the representation of the solution, therefore, one expects some deterioration in the wave speed. This is indeed the case. Figure 14 compares the case 3a (third degree shape functions, weight functions derived from the shape function) with a case 5a where one of the weight functions is replaced by the requirement of continuity of the second derivative. Figure 15 gives the same comparison for the case 3b constant weight functions with case 5b. One has indeed a deterioration of the result for shorter waves. (In these curves the maximum value of μh is π. A comparison with Figure 11 shows that this procedure is indeed much better than for first degree shape functions.

In third degree splines, the second derivative is defined everywhere. It is then possible to use a collocation method, in which the differential equation is satisfied for the grid points $y = kh$ (case 7). The results agree with curve c in Figure 1. A comparison of different third degree formulae is shown in Figure 16. Cases 5a and 5b

(third degree splines give nearly identical results which are fairly acceptable). A collocation method based on third degree splines is not an improvement over first degree functions.

The results for second degree shape functions, cases 4a and 4b, are shown in Figure 17. In Figures 18 and 19 the cases 4a and 4b (second degree shape functions) are compared with cases 6a and 6b (second degree splines). Second degree splines are very inferior.

Figure 20 shows a comparison between the results for third degree shape functions (case 3b) and a quadratic shape function (case 6b). The comparison is between a moderately good case for third degree shape functions and the best case for second degree shape functions. In the overall picture the two methods are about equivalent, for short waves the second degree method gives better results. However, for long waves which are the more important ones, the third degree method gives a better approximation to the wave speed ($\nu/\mu = 1$) over a wider range of values μh. The amount of labor in solving the system of ordinary differential equations is the same for third and second degree shape functions. One concludes that the use of third degree shape functions, or the use of third degree splines, is preferable. However, this conclusion is not generally correct because it disregards the labor to set up the system. This seems to hold in particular for elliptic equations.

SECTION V

REMARKS ABOUT THE NUMERICAL SOLUTION OF THE SYSTEMS OF DIFFERENTIAL EQUATIONS DERIVED IN SECTION II

The discussion of Section IV presuppose that one solves the systems of differential equations which arise by various methods of semidiscretization perfectly. In reality, one applies also for this purpose some numerical approach. We shall see that the error incurred in the solution of the differential equations for the time dependence combine with the errors due to the semidiscretization. Under favorable conditions they may cancel each other. These phenomena will be discussed later. At the moment we discuss the application of finite difference techniques to the systems of differential equation s of **Section** II. One might think of some carefully written code with provisions for error control. Most of these methods make use of information generated at several preceeding points in time.

One notices that all formulae based on the finite element approach fail to give explicit expressions for the time derivatives $d^2\phi_k/dt^2$ or $d^2\phi_k'/dt^2$. To obtain these derivatives in terms of the value of the functions ϕ_k and ϕ_k', one must solve a system of linear equations. For one space dimension this task is not too time consuming. One then obtains a banded matrix and the system of equations can be solved very efficiently. The problem becomes cumbersome for more than one space dimension. For two space dimensions, one obtains a block tridiagonal matrix where the size of the blocks depends upon the number of grid points in one space direction, thus the individual matrices may be rather large. An iterative solution of these systems may be possible in the cases 1 and 2 (Eqs. (11) and (13)) and also in Eqs. (37) and (38). This amounts to replacing the denominators (which come from the time derivatives) by the series

$$(1-\alpha\sin^2(\mu h/2))^{-1} = 1+\alpha\sin^2(\mu h/2) + [\alpha\sin^2(\mu h/2)]^2 + \cdots$$

The series converges for the values of α which occur in these formulae. The author has not explored the question for more than one space dimension.

One is tempted to use implicit methods for the solution of the differential equations. Implicit methods are particularly useful for stiff differential equations; that is, for equations in which the matrix governing the linearized system (after it has been resolved with respect to the derivatives) has eigenvalues with large negative real parts. If one solves such systems by the Runge Kutta method, or one of the predictor corrector methods, then the step in the t direction is limited for reasons of stability to about the reciprocal of the largest eigenvalue. A suitable implicit method removes this limitation. The step which is automatically chosen by the routine on the basis of an accuracy check is determined by the requirement that within the current step the solution can be represented with a desired accuracy by a polynomial of a chosen order. This allows the method to proceed in large steps in regions where the solution is smooth. Unfortunately, these preconditions are not met under the present circumstances. The eigenvalues are imaginary. One cannot expect to obtain smooth functions $\phi_k(t)$ since the eigensolutions pertaining to large eigenvalues (short waves) do not die out.

Here one might make the following argument: If a small step is needed in the y direction to express the solution with sufficient accuracy, then one needs in principle a corresponding small step in the t direction. Otherwise, one will disregard information which has been considered as

essential in the initial conditions. From the point of view of accuracy, one cannot permit in implicit methods a time step which is much larger than the one permissible in explicit procedures, but the same limitation will probably be encountered also from the point of view of stability of the numerical integration procedure.

The implicit method requires the solution of a linear system of equations at each time step, even if the differential equation gives the derivatives explicitly. But, under these circumstances it is not necessary to have the derivatives in an explicit form. To show this in a schematic manner, let us consider the system of first order equations

$$L\dot{y} + My = r$$

where y and r are vectors and L and M are matrices. In an implicit scheme one ultimately arrives at an equation

$$y(t+h) + \text{const}\ h\ \dot{y}(t+h) = u$$

where u is a known vector determined from the values of y and $\dot{y}$ at preceeding points in time and on the value of r(t). The specific form of the equation depends upon the integration method. One obtains by multiplying this equation from the left with L

$$Ly(t+h) + \text{const}\ h\ L\dot{y}(t+h) = Lu$$

$$(L - \text{const}\ h\ M)y(t+h) = Lu - \text{const}\ h\ r$$

In order to obtain y(t + h), one must therefore solve a system with the matrix L - const. h M. If the derivatives are explicitly available, then L is replaced by the identity

matrix. If one wants to use this idea, it may be necessary to rewrite existing routines for the implicit integration of differential equations.

According to an idea of Soliman et al., the inversion of the operator L in multidimensional problems is facilitated if the weight functions and the shape functions have a special form. In the two dimensional problems, one will introduce a two-dimensional grid (say a rectangular grid) with grid points characterized by two subscripts. To the point i,k there belongs a shape function $N_{i,k}$ and also a weight function. Assume that the shape function and the weight functions appear in the form

$$N_{i,k}(y,t) = f_1(y) f_2(t)$$

The inversion of the matrix L can then be reduced to the repeated solutions of one-dimensional problems. The price for this simplification is the restriction in the choice of shape and weight functions. Practically only very simple shapes are possible; for instance, piecewise linear functions in a rectangular grid. Even then, a direction inversion of the matrix (L - const hM) is not possible. If the second term is small enough, an additional iteration would be applicable. This restriction of flexibility (which to some extent contradicts the basic philosophy of the finite element approach) may be worthwhile for multidimensional problems where the solution of a large system of equations is a very time consuming element.

SECTION VI
TIME DEPENDENCE

In the preceding sections the original partial differential equation has been reduced to a system of ordinary differential equations (in our particular example with constant coefficients). Appendix IV derives the familiar fact that the results of the treatment of this system by finite difference or finite element methods remains unchanged if one first makes a transformation which brings the system into its diagonal form. It is therefore possible to consider one component at a time. The following discussions are therefore restricted to the equation

$$d^2u/dt^2 + \nu^2 u = 0 \tag{54}$$

where u is a scalar quantity; the constant ν stands for one of the values of ν computed in Section III as a function of the reciprocal of the wave lenght (except for a factor of 2π). In practice one does not make such a diagonalizing transformation, the values of ν never appear, and the shape functions and weight functions used in solving the system are independent of ν.

In Section II shape functions for the y direction have been used. The parameters upon which they depend are functions of time, which are now represented individually by shape functions of the same character. The scalars u in Eq. (54) are linear combinations of such parameters. One arrives at shape functions in the y,t plane of considerable complexity in spite of the fact that the discussion is carried out for one-dimensional problems. Familiar forms are bilinear, biquadratic and bicubic shape and weight functions in a rectangular grid.

We distinguish between steps h_s and h_t in the space and time directions respectively. The stability of the time integration depends upon $\nu^2 h_t^{\,2}$. The maximum of ν^2 which occurs in a specific problem depends upon the step h_s in the space direction. Sometimes it may be desirable to have a method which is stable even if νh_t is large.

Hyperbolic problems, by their very nature, are initial value problems; at an initial value of t the values of u and du/dt in Eq. (54) are prescribed. The solutions can therefore be obtained by a marching procedure. We shall restrict ourselves to methods which can be interpreted in this manner.

The application of finite difference methods has been discussed in general terms in Section IV. Here we discuss a specific approach which is sometimes advocated. One writes Eq. (54) as a system of first order equations

$$\begin{aligned} \frac{du}{dt} - \dot{u} &= 0 \\ \frac{d\dot{u}}{dt} + \nu^2 u &= 0 \end{aligned} \tag{55}$$

We use dots to indicate differentiation with respect to time. In Section II differentiations with respect to y/h_s have been denoted by prime.

We use equidistant points in time

$$t_k = kh_t$$

and set

$$u(kh_t) = u_k$$
$$u^{\cdot}(kh_t) = u^{\cdot}_k$$

Let θ be a constant; $0 \leq \theta \leq 1$. Then one obtains in a familiar manner

$$(u_{k+1} - u_k) - h_t\,(\theta u^{\cdot}_k + (1-\theta) u^{\cdot}_{k+1}) = 0$$
$$(u^{\cdot}_{k+1} - u^{\cdot}_k) + \nu^2 h_t\,(\theta u_k + (1-\theta) u_{k+1}) = 0$$

or

$$\begin{bmatrix} 1 & -h_t(1-\theta) \\ \nu^2 h_t(1-\theta) & 1 \end{bmatrix} \begin{bmatrix} u_{k+1} \\ u^{\cdot}_{k+1} \end{bmatrix} + \begin{bmatrix} -1 & -h_t\theta \\ \nu^2 h_t\theta & -1 \end{bmatrix} \begin{bmatrix} u_k \\ u^{\cdot}_k \end{bmatrix} = 0 \tag{56}$$

From this system one finds u_{k+1} and $u^{\cdot}_{k+1}$ in terms of u_k and $u^{\cdot}_k$. To study the stability one introduces an amplification factor ρ.

$$\begin{bmatrix} u_{k+1} \\ u^{\cdot}_{k+1} \end{bmatrix} = \rho \begin{bmatrix} u_k \\ u^{\cdot}_k \end{bmatrix} \tag{57}$$

This leads to the equation

$$\begin{vmatrix} -1+\rho & -h_t[\theta + \rho(1-\theta)] \\ \nu^2 h_t[\theta + \rho(1-\theta)] & -1+\rho \end{vmatrix} = 0 \tag{58}$$

Hence $$(\rho - 1)^2 + \nu^2 h_t^2 [\theta + \rho(1-\theta)]^2 = 0$$

$$\rho = \frac{1 \pm i\,(\nu h_t)\,\theta}{1 \mp i\,(\nu h_t)(1-\theta)}$$

One has

$$|\varrho| = \left[\frac{1 + (\nu h_t)^2 \theta^2}{1 + (\nu h_t)^2 (1-\theta)^2}\right]^{1/2} \tag{59}$$

The method is stable for $\theta \leq 1/2$. One also has

$$\log \varrho = \frac{1}{2} \log \frac{1 + (\nu h_t)^2 \theta^2}{1 + (\nu h_t)^2 (1-\theta)^2} \pm i \left[\operatorname{arctg}(\nu h_t \theta) + \operatorname{arctg}(\nu h_t (1-\theta))\right] \tag{60}$$

A single wave, defined by its value at the grid points is given by

$$\begin{aligned}
\phi &= \cos(\mu y)\, Re(\varrho^k) \\
&= \cos(\mu y)\, Re\left\{\exp\left(t_k \frac{1}{h_t} \log \varrho\right)\right\} \\
&= \left[\exp\left(t_k \frac{1}{h_t} \log|\varrho|\right)\right] \cos(\mu y) \cos\left\{t_k \frac{1}{h_t}\left[\operatorname{arctg}(\nu h_t \theta) + \operatorname{arctg}(\nu h_t (1-\theta))\right]\right\}
\end{aligned} \tag{61}$$

If there were no phase error the factor of t_k in the last term, namely

$$\frac{1}{h_t}\left[\operatorname{arctg}(\nu h_t \theta) + \operatorname{arctg}(\nu h_t (1-\theta))\right]$$

would be equal to μ.

Depending upon the specific form of the differential equations derived in Section II and upon the choice of h_s one has different relations between ν and μ. Ultimately, one is interested in the relation between μ and quantities characterizing the amplification factor. For some values of h_s/h_t and for some values of μ a cancellation of the discretization errors may occur.

The relations between μ and ν for cases 1 and 2, Eqs. (36), (37), and (38) can be written as:

$$\nu^2 h_s^2 = \frac{4 \sin^2(\mu h_s/4)}{1-\alpha \sin^2(\mu h_s/4)}$$

with $\alpha = 0$, $2/3$, and $1/2$, respectively. For these values of α and for different values of the Courant number $\beta = h_t/h_s$, the following expression is shown in Figs. 21, 22, and 23.

$$\frac{1}{\mu h_t}\left[\operatorname{arctg}(\nu h_t \theta) + \operatorname{arctg}(\nu h_t(1-\theta))\right] = \frac{1}{\mu h_s} \cdot \frac{1}{\beta}\left[2 \operatorname{arctg} \frac{\nu h_s}{2} \beta\right]$$

This expression arises from Eq. (61) for $\beta = 1/2$. In this case $|\rho| = 1$ and one has no damping. Ideally this expression ought to be 1 for all values of μ_s. The phase errors which arise are by no means small. The curves shown in Fig. 11 supplement these figures. They give the results for Courant number 0. In all cases Courant number 1/2 gives about the best results, particularly in conjunction with method 2b. The results deteriorate considerably with increasing Courant number.

Next we study cases in which the discretization applied in Section II for the space direction are used also for the time direction. We restrict ourselves to cases 1 through 3. In actual computations $\phi(y_\ell, t_{k-1})$, $\phi(y_\ell, t_k)$ and in cases 3 also $\dot\phi(y_\ell, t_{k-1})$ and $\dot\phi(y_\ell, t_k)$ are known. Here t_k refers to the time for which the computation has just been completed, ℓ ranges over all stations in the y-direction. The procedure then computed $\phi(y_\ell, t_{k+1})$ and in cases 3, also $\dot\phi(y_\ell, t_{k+1})$. According to the observations made at the beginning of this section it suffices if one studies Eq. (54). Then one determines u_{k+1} from u_{k-1} and u_k and, in cases 3, u_k+1 and $\dot u_{k+1}$ from u_{k-1}, $\dot u_{k-1}$, u_k, and $\dot u_k$. One obtains indeed a

marching procedure.

Case 2a gives

$$\nu^2[u_k + \frac{1}{6}(u_{k+1} - 2u_k + u_{k-1})] + h_t^{-2}(u_{k+1} - 2u_k + u_{k-1}) = 0$$

In the present context ν is given as a function of μh_s. The formulae are found in Section III. Particular solutions of this equation are found by the hypothesis

$$u_k = \exp(i\nu_1 k h_t)$$

where ν_1 is unknown. One then obtains

$$(\nu h_t)^2\left[1 - \frac{2}{3}\sin^2\left(\frac{\nu_1 h_t}{2}\right)\right] - 4\sin^2\left(\frac{\nu_1 h_t}{2}\right) = 0$$

Hence, for case 2a, after substitution of Eq. (37)

$$\left(\frac{h_t}{h_s}\right)^2 \frac{4\sin^2\left(\frac{\mu h_s}{2}\right)}{1 - \frac{2}{3}\sin^2\left(\frac{\mu h_s}{2}\right)} = \frac{4\sin^2\left(\frac{\nu_1 h_t}{2}\right)}{1 - \frac{2}{3}\sin^2\left(\frac{\nu_1 h_t}{2}\right)} \tag{62}$$

For $h_t/h_s = 1$ (Courant number 1) one obviously has

$$\nu_1 = \pm\mu$$

This means that the wave speed is correct. The errors due to the discretization for the t direction cancel the errors of the y discretization. In the interval $0 \leq \nu h_t \leq \pi$ the expression on the right of Eq. (62) is a monotonically increasing function of $\sin^2(\nu_1 h_t/2)$, its maxium is 12. If the left hand side exceeds this value, then the value of ν_1 for which this equation is solved cannot be real. From the two conjugate complex solutions which will then appear, one will give waves whose amplitudes increase with time.

For $h_t/h_s \leq 1$ one has a real value of ν_1 for each choice of μ; the method is stable for Courant numbers smaller than 1. The cancellation of discretization errors for Courant number 1 has no counterpart in elliptic equations.

The procedure for third degree shape functions can be carried out in analogous manner. However, one encounters a significant difference. Consider case 3a. The relation between μ and ν is given by Eq. (39). It has been obtained from Eq. (16) by setting

$$\phi_{k+1} = \phi_k \exp(i\mu h_s).$$

For the present discussion is it preferable if we write

$$\phi_{k+1} = \phi_k \rho \tag{63}$$

where ρ is some point on the unit circle in the complex plane. Then one obtains instead of Eq. (39)

$$\left| \begin{bmatrix} -\frac{6}{5}(\rho - 2 + \rho^{-1}) & \frac{1}{10}(\rho - \rho^{-1}) \\ -\frac{1}{10}(\rho - \rho^{-1}) & \frac{1}{5} - \frac{1}{30}(\rho - 2 + \rho^{-1}) \end{bmatrix} - \nu^2 h_s^2 \begin{bmatrix} 1 + \frac{9}{70}(\rho - 2 + \rho^{-1}) & -\frac{13}{420}(\rho - \rho^{-1}) \\ \frac{13}{420}(\rho - \rho^{-1}) & \frac{1}{210} - \frac{1}{140}(\rho - 2\rho + \rho^{-1}) \end{bmatrix} \right| = 0 \tag{64}$$

For ρ on the unit circle, this equation gives real values of $\nu^2 h_s$. Now we apply a discretization analogous to case 3a to Eq. (54). The results can be written down, by making appropriate modifications in Eq. (16).

$$h_t^{-2}\Big\{\frac{6}{5}(u_{k+1}-2u_k+u_{k-1})-\frac{1}{10}(\dot u_{k+1}-\dot u_{k-1})\Big\}+\nu^2\Big\{u_k+\frac{9}{70}(u_{k+1}-2u_k+u_{k-1})-\frac{13}{420}(\dot u_{k+1}-\dot u_{k-1})$$
$$=0$$

$$h_t^{-2}\Big\{\frac{1}{10}(u_{k+1}-u_{k-1})-\Big[\frac{1}{5}\dot u_k-\frac{1}{30}(\dot u_{k+1}-2\dot u_k+\dot u_{k-1})\Big]^2+\nu^2\Big\{\frac{13}{420}(u_{k+1}-u_{k-1})+\frac{1}{210}\dot u_k-\frac{1}{140}(\dot u_{k+1}$$
$$-2\dot u_k+\dot u_{k-1}\Big\}=0$$

Setting

$$u_{k+1} = u_k \cdot \rho_1, \quad \dot u_{k+1} = \dot u_k \cdot \rho_1$$

and changing all signs one arrives at

$$\left|\begin{bmatrix} -\frac{6}{5}(\rho_1-2+\rho_1^{-1}) & \frac{1}{10}(\rho_1-\rho_1^{-1}) \\ -\frac{1}{10}(\rho_1-\rho_1^{-1}) & \frac{1}{5}-\frac{1}{30}(\rho_1-2+\rho_1^{-1}) \end{bmatrix} - \nu^2 h_t^2 \begin{bmatrix} 1+\frac{9}{70}(\rho_1-2+\rho_1^{-1}) & \\ \frac{13}{420}(\rho_1-\rho_1^{-1}) & \frac{1}{210}-\frac{1}{140}(\rho_1-2+\rho_1^{-1}) \end{bmatrix}\right| = 0 \tag{65}$$

This equation must be solved for ρ_1. It can be rewritten in terms of an unknown

$$v = \rho_1^{1/2} - \rho_1^{-1/2} \tag{66}$$

One has specifically

$$\rho_1 - 2 + \rho_1^{-1} = v^2$$

$$\rho_1 - \rho_1^{-1} = (\rho_1^{1/2}+\rho_1^{-1/2})(\rho_1^{1/2}-\rho_1^{-1/2}) = v\,(v^2+4)^{1/2}$$

Thus, from Eq. (65)

$$\left|\begin{bmatrix} -\frac{6}{5}v^2 & \frac{1}{10}v(v^2+4)^{1/2} \\ -\frac{1}{10}v(v^2+4)^{1/2} & \frac{1}{5}-\frac{1}{30}v^2 \end{bmatrix} - \nu^2 h_t^2 \begin{bmatrix} 1+\frac{9}{70}v^2 & -\frac{13}{420}v(v^2+4)^{1/2} \\ \frac{13}{420}v(v^2+4)^{1/2} & \frac{1}{210}-\frac{1}{140}v^2 \end{bmatrix}\right| = 0$$

This is a quadratic equation for v^2. Eq. (65) can be obtained from Eg. (64) by replacing ρ by ρ_1 and h_s by h_t. For a further discussion it sufficies if one sets $h_s = h_t$. Then one has obviously

$$\rho_1 = \rho = \exp(i\mu h_s) \tag{67}$$

These values of ρ_1 lie on the unit circle in the complex plane. The value of v^2 pertaining to these solutions is

$$v^2 = -4 \sin^2\left(\frac{\mu h_s}{2}\right)$$

But there exists a second solution for v^2. We shall see that among values of ρ_1 belonging to this second solution there is always one which gives an unstable solution. It follows from Eq. (65) that the product of the two roots for ρ_1 is always 1. All nonincreasing particular solutions must have values of ρ_1 lying on the unit circle, for there belongs to each root which lies inside the unit circle another one outside. If one chooses ρ_1 as a point of the unit circle then one can evaluate Eq. (65) for $\nu^2 h_t^{\,2}$. (This is the evaluation which led to Fig. 12 curve a, but now we consider μh_t and νh_t as coordinates.) To each value of μh_t one obtains two values of νh_t, each a monotonic function of μh_t; these two functions do not overlap. It follows that for these curves (which exhaust all possibilities for ρ_1 being on the unit circle) one obtains for each choice of νh_t one value of μh_s (and consequently one value of v^2). The slope of these curves does not vanish, accordingly there are no double roots. It follows that the second root for v^2 gives values of ρ_1 off the unit circle, one of them pertains to an unstable solution. The same argument can, of course, be made if $h_t/h_s \neq 1$.

This observation can be connected to a phenomenon which arises if one solves ordinary differential equations by integration techniques in which one uses information generated for preceding points in time. One then deals in principle

with difference operators which are of a higher order than the differential operator occurring in the differential equation that is to be solved. This procedure introduces spurious particular solutions. In order for such a scheme to be stable, these spurious solutions must be damped. If one uses third degree shape functions in a manner analogous to case 2a to solve the differential equation for the time dependence, then one uses information for u_k, $\dot{u}_k$, u_{k-1}, and $\dot{u}_{k-1}$. To define an initial value problem for the differential equations the values of u_k and $\dot{u}_k$ are sufficient. One recognizes that this form of discretization will introduce spurious particular solutions and because of the complete symmetry which exists between the positive and the negative time direction, there exists for each solution which is damped another one which is excited (that is, damped in the negative time direction).

Because of this observation, higher order elements of the character discussed in cases 2 and 3 must be rejected.

For first order elements this phenomenon does not occur, but even then one might have reservations because of the choice of the weight functions. In a marching procedure one considers the solution as known up to a point t_k, and continues it through the interval between t_k and t_{k+1}. This step has no influence on the solution in the preceding interval. However, the weights used in cases 2 and 3 take into account the residuals between the points t_{k-1} and t_k as well as those between the points t_k and t_{k+1}. The solution in the interval between t_k and t_{k+1} is chosen in such a manner that its residual counteracts the effect of the residual in the preceding interval, over which one has no control. One observes, on the other hand,

that according to the above stability analysis this has no obvious undesirable effect. For Courant number 1 the methods even give a cancellation of the discretization errors in time and space directions.

Let us now consider procedures for which this objection does not hold.

First we discuss shape functions which are piecewise linear in space as well as in time. In the choice of the weights for the space direction one will observe the symmetry which exists between positive and negative values of y. Regarding the time direction a corresponding symmetry is not required. For higher order elements it is even undesirable (because of the occurrence of spurious solutions, as we have seen above). Specifically we choose, for piecewise linear shape functions

$$w=1, \quad \text{for } kh_t - \varepsilon \le t \le (k+1)h_t - \varepsilon, \quad \varepsilon > 0, \ \varepsilon \to 0$$

The weight function straddles the point t_k but not the point t_{k+1}. This is necessary because one will have a jump of $\dot{u}$ at the point t_k which generates a δ - function when one forms d^2y/dt^2. One has

$$\int_{t=kh_t-\varepsilon}^{(k+1)h_t-\varepsilon} w(t)\frac{d^2u}{dt^2}dt = \int_{t=kh_t-\varepsilon}^{(k+1)h_t-\varepsilon} \frac{d^2u}{dt^2}dt = \frac{du}{dt}\Bigg|_{t=kh_t-\varepsilon}^{t=(k+1)h_t-\varepsilon}$$

$$= h_t^{-1}(u_{k+1}-u_k) - h_t^{-1}(u_k-u_{k-1}) = h_t^{-1}(u_{k+1}-2u_k+u_{k-1})$$

Furthermore

$$\lim_{\varepsilon\to 0}\int_{t=kh_t-\varepsilon}^{(k+1)h_t-\varepsilon} u\,dt = \int_{t=kh_t}^{t=(k+1)h_t} u\,dt = \frac{u_{k+1}+u_k}{2}h_t$$

One thus obtains

$$u_{k+1} - 2u_k + u_{k-1} + \nu^2 h_t^2 \, \frac{u_{k+1}+u_k}{2} = 0 \tag{68}$$

Again, we introduce an amplification factor

$$u_{k+1} = \rho \, u_k \tag{69}$$

This leads to

$$\rho^2\left(1 + \frac{1}{2}(\nu h_t)^2\right) - \left(2 - \frac{1}{2}(\nu h_t)^2\right)\rho + 1 = 0 \tag{70}$$

$$\rho = \frac{1}{1+\frac{1}{2}(\nu h_t)^2}\left[1 - \frac{1}{4}(\nu h_t)^2 \pm i\nu h_t\left[1 - \frac{1}{16}(\nu h_t)^2\right]^{\frac{1}{2}}\right] \tag{71}$$

For small values of νh_t the roots are conjugate complex. The absolute value can be computed from the last formula. It can be obtained more simply by a comparison of the first and the last term in Eq. (70). One obtains

$$|\rho| = \left|1 + \frac{1}{2}(\nu h_t)^2\right|^{-1/2} \tag{72}$$

A double root is obtained for $(\nu h_t)^2 = 16$. This gives

$$\rho = -\frac{1}{3}$$

For $(\nu h_t)^2 > 16$ one obtains two real values of ρ. The particular solutions are always damped if $|\rho| < 1$.

We examine for which values of νh_t one will have $|\rho| = 1$. One obtains from Eq. (70)

and

$$(\nu h_t)^2 = 0 \quad \text{for } \rho = 1$$

$$(\nu h_t)^2 = \infty \quad \text{for } \rho = -1$$

The particular solutions are damped for all finite values of νh_t. As νh_t tends to infinity one obtains particular solutions which alternate between positive and negative with little change of amplitude. Fig. 24 shows $\exp(i\mu h_t)$ and a curve $\rho(\nu h_t)$ in the complex plane for $0 \leq \mu h_t \leq \pi$. Corresponding points are connected with each other. The useful range for values of μh_t is rather limited to about $0 \leq \mu h_t \leq \pi/2$. Beyond this point one obtains strong damping and a considerable phase error. The method can be used even if μh_t is large, because such waves do not create an instability, but the contributions of waves for large values of μh_t are meaningless.

Now we study again Eq. (54) but represent the unknown in the interval from k to k + 1 by third degree polynomials They are determined by the known quantities u_k and $\dot{u}_k$ and by the unknown quantities u_{k+1} and $\dot{u}_{k+1}$. By this characterization continuity of the first derivates at the grid points is guaranteed. The functions u represent the coefficients of the eigenfunctions which arise from a representation of ϕ (in the original partial differential equation) along lines t = constant in terms of piecewise third degree polynomials in y. They are therefore expressible in terms of ϕ and ϕ_y at the grid points. The continuity of the derivatives $\dot{u}$ therefore implies continuity of ϕ_t and ϕ_{yt} not only at the grid points but along lines t = const. If the function ϕ_t is given along a line t = constant (which is the case in a properly set initial value problem) then one can determine from it also ϕ_{yt}; it does not contradict the notion of a properly formulated initial value problem, if we initially assign the value of $\dot{u}$. If one takes triangular elements and third-degree polynomials (with the powers of y and t

counted together) then one has agreement of the gradient in adjacent intervals only at the grid points. In the present formulation we have piecewise bicubic elements and then continuity of the gradient is guaranteed everywhere along the element boundaries.

The weight functions are to be applied only to the interval from k to k + 1. No overlap over the grid point is needed, because of the continuity of the first derivatives. In each interval we have two unknowns, u_{k+1} and $\dot{u}_{k+1}$. One therefore needs two weight functions. We choose

$$
\begin{aligned}
w &= 1 \text{ for } t_k \leq t \leq t_k + c\, h_t \quad (0 \leq c \leq 1) \text{ and} \\
w &= 1 \text{ for } t_k + c\, h_t \leq t \leq t_{k+1}
\end{aligned}
\tag{73}
$$

or equivalently,

$$w = 1 \text{ for } t_k < t < t_{k+1}$$

$$w = 1 \text{ for } t_k < t < t_k + ch$$

In representing the space dependence of ϕ we have also considered examples in which the weight functions were formed by linear combinations of the elemental shape functions. Something similar can be done here. One could for instance choose

$$w = N_1 + N_3$$

and

$$w = N_2 + N_4$$

where the functions N_i are given by Eq. (6) with Δy replaced by Δt. The first of these weight functions is 1.

This possibility has not been discussed. The results would probably be of the same character as those for the weight functions defined above. Let

$$\Delta t = t - t_k.$$

Then, using Eq. (6), one has the following representation for u in the interval between t_k and t_{k+1}.

$$u(t) = u_k\left[1 - 3\left(\frac{\Delta t}{h_t}\right)^2 + 2\left(\frac{\Delta t}{h_t}\right)^3\right] + u_{k+1}\left[3\left(\frac{\Delta t}{h_t}\right)^2 - 2\left(\frac{\Delta t}{h_t}\right)^3\right]$$
$$+ \dot{u}_k h_t\left[\frac{\Delta t}{h_t} - 2\left(\frac{\Delta t}{h_t}\right)^2 + \left(\frac{\Delta t}{h_t}\right)^3\right] + \dot{u}_{k+1} h_t\left[-\left(\frac{\Delta t}{h_t}\right)^2 + \left(\frac{\Delta t}{h_t}\right)^3\right]$$

$$\frac{d^2u}{dt^2} = u_k h_t^{-2}\left[-6 + 12\left(\frac{\Delta t}{h}\right)\right] + u_{k+1} h_t^{-2}\left[6 - 12\left(\frac{\Delta t}{h}\right)\right]$$
$$+ \dot{u}_k h_t^{-1}\left[-4 + 6\left(\frac{\Delta t}{h}\right)\right] + \dot{u}_{k+1} h_t^{-1}\left[-2 + 6\left(\frac{\Delta t}{h}\right)\right]$$

Substituting these expressions into Eq. (54), multiplying by the weight function 1 and integrating over $\frac{\Delta t}{h_t}$ from 0 to c, one obtains after multiplication by h_t^2

$$\begin{aligned}
&\{u_k[-6c + 6c^2] + u_{k+1}[6c - 6c^2] \\
&+ \dot{u}_k h_t[-4c + 3c^2] + \dot{u}_{k+1} h[-2c + 3c^2]\} \\
&+ \nu^2 h_t^2\{u_k[c - c^3 + \tfrac{1}{2}c^4] + u_{k+1}[c^3 - \tfrac{1}{2}c^4] \\
&\quad + \dot{u}_k h_t[\tfrac{1}{2}c^2 - \tfrac{2}{3}c^3 + \tfrac{1}{4}c^4] + \dot{u}_{k+1} h_t[-\tfrac{1}{3}c^3 + \tfrac{1}{4}c^4]\} \\
&= 0
\end{aligned} \tag{74}$$

$$-u_k^{\cdot} h_t + u_{k+1}^{\cdot} h_t \tag{74}$$

Continued

$$+ \nu^2 h_t^2 \left\{ \tfrac{1}{2} u_k + \tfrac{1}{2} u_{k+1} + \tfrac{1}{12} u_k^{\cdot} h_t - \tfrac{1}{12} u_k^{\cdot} h_t \right\} = 0$$

The second equation is obtained by setting $c = 1$. Let

$$\nu^2 h_t^2 = \lambda .$$

Then one obtains the following system of equations which connects the vectors $(u_{k+1}, u_{k+1}^{\cdot})^+$ and $(u_k, u_k^{\cdot})^+$.

$$\begin{bmatrix} \frac{\lambda}{2} & (-1 + \frac{\lambda}{12}) \\ a_1 + \lambda b_1 & a_2 + \lambda b_2 \end{bmatrix} \begin{bmatrix} u_k \\ u_k^{\cdot} h_t \end{bmatrix} + \begin{bmatrix} \frac{\lambda}{2} & (1 - \frac{\lambda}{12}) \\ c_1 + \lambda d_1 & c_2 + \lambda d_2 \end{bmatrix} \begin{bmatrix} u_{k+1} \\ u_{k+1}^{\cdot} h_t \end{bmatrix} = 0 \tag{75}$$

with

$$\begin{aligned} a_1 &= -6c + 6c^2 & c_1 &= 6c - 6c^2 \\ a_2 &= -4c + 3c^2 & c_2 &= -2c + 3c^2 \\ b_1 &= c - c^3 + \tfrac{1}{2}c^4 & d_1 &= c^3 - \tfrac{1}{2}c^4 \\ b_2 &= \tfrac{1}{2}c^2 - \tfrac{2}{3}c^3 + \tfrac{1}{4}c^4 & d_2 &= -\tfrac{1}{3}c^3 + \tfrac{1}{4}c^4 \end{aligned} \tag{76}$$

The amplification ratio ρ defined by

$$\begin{bmatrix} u_{k+1} \\ \dot{u}_{k+1} h_t \end{bmatrix} = \rho \begin{bmatrix} u_k \\ \dot{u}_k h_t \end{bmatrix}$$

is determined by the following equation

$$\left| \begin{pmatrix} \frac{\lambda}{2} & (-1 + \frac{\lambda}{12}) \\ (a_1 + \lambda b_1) & (a_2 + \lambda b_2) \end{pmatrix} + \rho \begin{pmatrix} \frac{\lambda}{2} & (1 - \frac{\lambda}{12}) \\ (c_1 + \lambda d_1) & (c_2 + \lambda d_2) \end{pmatrix} \right| = 0 \qquad (77)$$

This quadratic equation for ρ must be solved for different values of λ and c. The amplification ratio has either two real or two conjugate complex roots. Eq. (77) evaluated in detail gives

$$\begin{aligned} c(1-c)\{ &[-6 + \lambda(-\tfrac{1}{2} + \tfrac{1}{2}c(1-c)) - \tfrac{\lambda^2 c^2}{12}]\rho^2 \\ &+ [12 + \lambda(-5 - c(1-c) + \tfrac{\lambda^2}{12}(1 + 4c(1-c))]\rho \qquad (78) \\ &+ [-6 + \lambda(-\tfrac{1}{2} + \tfrac{1}{2}c(1-c)) - \tfrac{\lambda^2(1-c)^2}{12}]\} = 0 \end{aligned}$$

This equation must be solved for different values of c and λ. The factor c(1-c) is of course unimportant. One notices that the coefficient of ρ^2 and the coefficient of the constant terms are interchanged if one replaces c by (1-c) and that the coefficient of ρ remains the same. Accordingly, if c is replaced by (1-c), then a solution of ρ is replaced by ρ^{-1}. For $c = \frac{1}{2}$ the product of the roots ρ is 1. In this case the solution is stable only if the roots are conjugate complex. The discussion for $c = \frac{1}{2}$ best goes back to the original determinant Eq. (77). Setting $c = \frac{1}{2}$ and taking twice the second row minus the first row, one generates the effect

of a weight function which is symmetric with respect to $\Delta t/h_t = \frac{1}{2}$. One obtains

$$\begin{vmatrix} \frac{\lambda}{2}(1+\rho) & (-1+\frac{\lambda}{12})(1-\rho) \\ (-3+\frac{5}{16}\lambda)(1-\rho) & (-\frac{3}{2}+\frac{1}{32}\lambda)(1+\rho) \end{vmatrix} = 0$$

Hence

$$\frac{(1-\rho)^2}{(1+\rho)^2} = -\frac{1}{4}\,\frac{\lambda(1-\frac{1}{48}\lambda)}{(1-\frac{\lambda}{12})(1-\frac{5}{48}\lambda)} \tag{79}$$

For λ small one obtains

$$\frac{1-\rho}{1+\rho} = \pm i\lambda^{1/2} = \pm i\nu h_t$$

and hence

$$\rho = 1 \pm i\nu h_t$$

(as expected). The right hand changes signs for $\lambda = 48$, $\lambda = 12$, and $\lambda = 9.6$.

ρ is complex for $0 \leq \lambda \leq 9.6$ and $12 \leq \lambda \leq 48$

ρ is real for $9.6 \leq \lambda \leq 12$ and $48 \leq \lambda$.

The values of $(\arg\rho)\lambda^{-1/2}$ versus $\lambda^{1/2}$ for $0 \leq \lambda^{1/2} = \nu h_t \leq 9.6$ are shown in Fig. 25. One sees that in this stable range the ideal value 1 is fairly well approximated. The method is unstable for $c = \frac{1}{2}$ and ρ real.

One has, from Eq. (78) for general c and two conjugate complex roots of ρ

$$|\rho|^2 = \frac{6 + \lambda\left(\frac{1}{2} - \frac{1}{2}c(1-c)\right) + \lambda^2 \frac{1}{12}(1-c)^2}{6 + \lambda\left(\frac{1}{2} - \frac{1}{2}c(1+c)\right) + \lambda^2 \frac{1}{12}c^2} \tag{80}$$

The first two terms in the numerator and in the denominator are the same; for $0 < c < 1$ they are positive. The last terms show that $|\rho| \leq 1$ for $c \geq \frac{1}{2}$. We restrict our attention to this region $1/2 \leq c \leq 1$. Fig. 26 shows for $c = 3/4$ the curves in the complex ρ plane which arise if λ is varied. They consist of arcs in the complex plane which are symmetric to the real axis. According to Eq. (80) these arcs lie within the unit circle if $1/2 < c < 1$. They end in branch points which lie on the real axis and are connected by pieces of the curve for which ρ is real. These portions may pass through the unit circle. This can happen only for $\rho = \pm 1$. One obtains from Eq. (76) for $\rho = 1$

$$\frac{\lambda^2}{2} c(1-c) - 6\lambda = 0.$$

Hence

$$\lambda = 0$$

and (81)

$$\lambda = \frac{12}{c(1-c)}$$

The first value represents the beginning of the curve. The last value is a second transition through the point 1. Only for $c = 1$ will this point lie at $\lambda = \infty$.

One obtains for $\rho = -1$

$$-\frac{\lambda^2}{6}[1 + c(1-c)] + \lambda[4 + 2c(1-c)] - 24 = 0$$

Hence

$$\lambda = 12$$

and

$$\lambda = \frac{12}{1+c(1-c)}$$

These points coincide for $c = 1$; that is, the ρ-curve reaches the unit circle at $\rho = -1$ but returns immediately. The arrows in Fig. 26 give the direction of increasing values of λ. The curve starts with $\lambda = 0$ at $\rho = 1$ and moves along two branches through the complex plane to a branch point on the negative real axis. It is continued along the real axis in the positive and negative directions. The branch extending in the negative direction passes through the unit circle. Both branches double back and meet at a branch point along the negative real axis. From there the curve is continued through the complex plane in two branches which meet at a branch point along the positive real axis. The final continuations follow the real axis in positive and negative directions. One of the branches passes through the unit circle (at $\rho = 1$). For $\lambda \to \infty$ these two branches end at points of the positive real axis. For $c = 1$ (Fig. 27) one of the endpoints lies on the unit circle.

According to the above analytical discussions the curves for different values of c have the same character. All have unstable regions ($|\rho| > 1$), except if $c = 1$. Stability for $c \neq 1$ can be guaranteed by limiting the Courant number ($\lambda \leq 12/(1 + c(1-c)$. For $c = 1/2$ this limit is $\lambda \leq 12$.

Notice that these values are close to $\lambda = (\nu h_t)^2 = \pi^2$. If one imposes such a limitation then the choice $c = 1/2$ seems to be preferable; according to Fig. 25 it gives a good approximation to the propagation velocity of different waves.

In comparison to linear shape functions the approximation by third degree shape functions is much better even for other values of c. This is seen by comparison of Figs. 26 and 27 with Figure 24, but as we mentioned, one must limit the Courant number.

It is clear from the outset that for Courant numbers that are considerably above 1 the solution of Eq. (54) will be inaccurate. The Green's function, which describes the propagation of errors, is an oscillating function. We have imposed an averaging procedure for the residuals without taking this property of the Green's function into account. One is therefore resigned to the fact that from a certain value of νh_t on the results will be meaningless. One hopes, however, that such solutions will vanish automatically, because they are damped. The above analysis shows that this is not the case for third degree shape functions except for $c = 1$. In this limiting case one has as one of the weight functions constant weight throughout the interval and as second weight function a δ function at the point t_{k+1}; this amounts to a combination of weighted residual method and a collocation method. Of course, a collocation method has much in common with a finite difference approach.

We notice in passing that an implicit method does not automatically lead to a stable procedure.

SECTION VII
A MORE COMPLICATED SAMPLE PROBLEM

In the preceding sections we have seen that in many methods stability can be achieved by limiting the Courant number. There are methods which are stable even for very large Courant numbers, but this entails a considerable loss of accuracy even for fairly long waves.

In a problem with variable coefficients it is possible to satisfy a Courant number limitation throughout the whole field provided that the discriminant which determines whether the problem is elliptic or hyperbolic is bounded away from zero. If the step in the space direction is fixed, one simply makes the time step sufficiently small. Of course, for certain parts of the field the Courant number will then be unnecessarily small and the progress in the t direction unnecessarily slow. At first glance it seems that this is no longer possible if this discriminant vanishes locally as it happens in the transonic problem. (In the present discussion which is restricted to hyperbolic equations, this can happen only at the boundary of the field.) We study a simply example of this kind.

Consider the partial differential equation

$$-y \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{83}$$

with boundary conditions

$$\begin{aligned} \phi &= 0 \qquad \text{for} \quad y = 0 \\ \phi &= 0 \qquad \text{for} \quad y = 1 \end{aligned} \tag{84}$$

The discriminant vanishes at the boundary y = 0 of the field. In a transonic problem, x is the downstream direction. It corresponds to t in the previous examples.

For a comparison we shall treat the wave equation also

$$-\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \tag{85}$$

with the same boundary conditions.

For simplicity a semidiscretized difference method with L + 1 equal intervals in the y direction (L intermediate points) is chosen. Then one has

$$h_y = \frac{1}{L+1} \tag{86}$$

Let

$$\phi_k(x) = \phi(x, kh_y) \tag{87}$$

The semidiscretized finite difference approximations to Eqs. (83) and (85) are, respectively

$$-kh_y \frac{\partial^2\phi_k}{\partial x^2} + h_y^{-2}(\phi_{k+1} - 2\phi_k + \phi_{k-1}) = 0 \tag{88}$$

(transonic equation)

and

$$-\frac{\partial^2\phi_k}{\partial x^2} + h_y^{-2}(\phi_{k+1} - 2\phi_k + \phi_{k-1}) = 0 \tag{89}$$

(wave equation)

Setting

$$\phi_k(x) = a_k \exp(i\nu x) \tag{90}$$

one obtains the following eigenvalue problems

$$\lambda k a_k + (a_{k+1} - 2a_k + a_{k-1}) = 0 \tag{91}$$

with

$$\lambda = \nu^2 h_y^3 \tag{92}$$

(transonic equation)

and

$$\lambda a_k + (a_{k+1} - 2a_k + a_{k-1}) = 0 \tag{93}$$

with $\lambda = \nu^2 h_y^2$ (94)

(wave equation)

The n^{th} eigenvector for the wave equation has the elements

$$a_k^{(n)} = \sin \frac{nk\pi}{L+1} \tag{95}$$

Then one has

$$\lambda^{(n)} = 4 \sin^2\left(\frac{n\pi}{L+1}\right) \tag{96}$$

The largest eigenvalue is obtained for n = L. If L is increased, then one obtains more eigenvalues but within the same interval $0 < \lambda^{(n)} < 4$.

For the transonic problem one has a generalized matrix eigenvalue problem

$$\left\{ \begin{bmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & & \cdot & & \\ & & & \cdot & & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{bmatrix} + \lambda \begin{bmatrix} 1 & & & & & \\ & 2 & & & & \\ & & 3 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & L-1 & \\ & & & & & L \end{bmatrix} \right\} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \cdot \\ \cdot \\ a_{L-1} \\ a_L \end{bmatrix} = 0 \tag{97}$$

This problem has been solved for L = 8, L = 12 and L = 20 by means of a routine available in the EISPACK library. The eigenvalues are listed in Table 1.

For the wave equation as well as for the transonic equation there exists a maximum eigenvalue ($\lambda = 2.4$ and $\lambda = 4$, respectively). For the integration with respect to time (in which the question of stability is decided) one must consider the value of ν which is connected to λ by Eqs. (92) and (94), respectively. In the preceding section we found as a stability limit for approximation of the time dependence by third order polynomials

$$(h_x \nu)^2 \sim 9.6$$

TABLE 1

EIGENVALUES FOR DIFFERENT VALUES OF L IN THE TRANSONIC CASE

L = 8	L = 12	L = 20
2.3880	2.3880	2.3880
1.0900	1.0900	1.0900
.70560	0.70566	.70566
.51691	.52166	.52166
.37290	.41367	.41376
.22900	.34015	.34284
.00672	.27464	.29267
.025700	.20644	.25531
	.13978	.22632
	.081353	.20215
	.036374	.17847
	.0085800	.15318
		.12699
		.10112
		.076658
		.054498
		.035403
		.019995
		.0087602
		.0020425

TABLE 2

THE VALUES OF a_k, k = 1...8 (FOR THE LARGEST EIGENVALUE OF THE TRANSONIC EQUATION

k	L = 8	L = 12	L = 20
1	1.00000	1.0000000	1.000000
2	-.38801545	-.38801545	-.038801545
3	.077142863	.77142863	.077142863
4	-.1035388	-.010353876	-.010353876
5	.0010502475	.0010502475	.0010502475
6	-.000085665	-.000085665	-.000085665
7	.0000058444	.0000058449	.0000058449
8	-.000000341697	-.00000034273133	-.00000034273133

TABLE 3

THE VALUES OF a_k, k = 1...8, FOR THE SECOND LARGEST EIGENVALUE OF THE TRANSONIC EQUATION

W	L = 8	L = 12	L = 20
1	-.85928478	-.85928475	-.85928475
2	-.78197087	-.78197083	-.78197083
3	1.000000	1.0000000	1.000000
4	-.48795330	-.48795340	-.48795340
5	.15152046	.1512073	.15152073
6	-.034773134	-.034773974	-.034773974
7	.0063442942	.0063478456	-.0063478456
8	-.0009411981	-.00096327439	-.00096327439

TABLE 4

THE VALUES OF a_k, k = 1...8 FOR THE FOURTH LARGEST EIGENVALUE

W	L = 8	L = 12	L = 20
1	.67082316	.67489092	.67489121
2	.99489036	.99772005	.99772028
3	.29041777	.27961324	.27961259
4	-.86441535	-.87608045	-.87608122
5	-.2319448	-.20371902	-.20371726
6	1.0000	1.0000000	1.0000
7	-.86952195	-.9262249	-.92622843
8	.40721511	.52975421	.52976194

To improve the accuracy one may choose a somewhat lower limit. Substituting Eqs. (92) and (94) into this condition and using the respective maximum values of λ one obtains

$$h_x^2 \cong 4 h_y^3 \qquad (98)$$

(transonic equation)

$$h_x^2 \cong 2.4 h_y^2 \qquad (99)$$

(wave equation)

Important are the powers of h_y (3 for the transonic equation, 2 for the wave equation). One must, of course, reduce h_x as one reduces h_y, but more strongly for the transonic problem than for the wave equation. But even here, a mesh for which the time integration converges can always be found.

One observes that in the transonic example the largest eigenvalues for different values of L agree nearly perfectly. This does not happen for the wave equation. Also the components $a_k^{(n)}$ of the eigenvectors (normalized by the requirement that the largest component of the vector with components $a_k^{(n)}$ be one) agree surprisingly well for different values of L. (See Table 2 for the largest eigenvalue, Table 3 for the second largest and Table 4 for the fourth largest.) For large values of k the components $a_k^{(n)}$ of these vectors become extremely small, therefore only the components for $k = 1$ to $k = 8$ are included in these tables. One sees that the effect of the eigenvectors pertaining to the largest eigenfunctions is restricted to the immediate vicinity of the parabolic line. As L is increased and the interval h_y becomes smaller, the region where the largest eigenvalues are of importance contracts because close agreement of the eigenvectors occurs if one compares the same values of k (not of $y = k\,h_y$)

This analysis shows that the Courant number limitations are not as serious as one might assume at a first glance. If the contributions belonging to the largest eigenvalues

should become unstable, then this instability will be confined to the immediate vicinity of the parabolic line $y = 0$.

The eigenvectors have been found by direct computation. The is probably the most practical way even if one does not use the EISPACK routine, which is somewhat oversophisticated for the present simple problem. The eigenfunctions can be expressed in terms of Bessel functions. This may be useful if one wants to understand their asymptotic properties, for instance, the fact that the largest eigenvalues are nearly identical.

The system Eq. (97) can be regarded as a three point recurrence relation

$$a_{k-1} + a_{k+1} + (k\lambda - 2)a_k = 0$$

It resembles the recurrence relation for Bessel functions

$$Z_{p-1}(x) + Z_{p+1}(x) - 2\frac{p}{x}Z_p = 0$$

where Z_p represents a linear combination of J_p and N_p with coefficients that are independent of p. We set

$$a_k = (-)^k \tilde{a}_k$$

Then one has

$$\tilde{a}_{k+1} + \tilde{a}_{k-1} - (k\lambda - 2)\tilde{a}_k = 0 \tag{100}$$

To identify the two relations one must postulate

$$k\lambda - 2 = \frac{2p}{x}$$

The order p must increase by 1 as k is increased by 1. Therefore,

$$\lambda = 2/x$$

and, using this result

$$p = k - 2/\lambda$$

Thus, one finds that

$$\tilde{a}_k = Z_{k-2/\lambda}(2/\lambda) \tag{101}$$

satisfies the recurrence relation Eq. (100). The eigenvalue λ is determined by the boundary conditions

$$\tilde{a}_0 = 0, \quad \tilde{a}_{L+1} = 0$$

Accordingly, one must determine λ and the linear combination between J_p and N_p in such a manner that

$$Z_{-2/\lambda}(2/\lambda) = 0 \tag{102}$$

and

$$Z_{L+1-2/\lambda}(2/\lambda) = 0 \tag{103}$$

If L is very large, then the order $L = 1 -(2/\lambda)$ is very large in comparison to the argument $(2/\lambda)$ of the Bessel functions. Bessel's equation reads

$$Z_p'' + \frac{1}{x} Z_p + \left(1 - \frac{p^2}{x^2}\right) Z_p = 0$$

If the order is very large in comparison to the argument, then (except for a constant) $Z_p = X^{\pm p}$. It then follows from Eq. (103) that Z is, in essence, given by $J_p = \text{const.}\ X^p$ while the contribution of N_p is very small (except for $k \sim L$). This explains why the eigenfunctions are so close to each other for small values of k.

SECTION VIII

A RATIONALE FOR THE CHOICE OF THE WEIGHT FUNCTIONS

So far we have chosen the weight functions on intuitive grounds and then examined their effect on the stability. In this section the attempt is made to provide for a more rational basis.

We ask for the best possible choice of a weight function for the differential equation Eq. (54) assuming that ν has a fixed value. As usual, we replace u by some approximation using shape functions which are related to certain data within the flow field; in our case these are the values of u and $\dot{u}$ at the grid points. The approximation so obtained is denoted by $\tilde{u}$. With weight functions w_1 and w_2, one obtains the following equations, from which the values of $\tilde{u}_k$ and $\dot{\tilde{u}}_k$ are determined.

$$\int_{t_k}^{t_{k+1}} \left(\frac{\partial^2 \tilde{u}}{\partial t^2} + \nu^2 \tilde{u}\right) w_i(\Delta t)\, dt = 0, \qquad i = 1, 2 \tag{104}$$

The specific choice, to be justified presently, is

$$w_1 = \cos(\nu \Delta t) \;;\; w_2 = \sin(\nu \Delta t) \tag{105}$$

Then one has the condition

$$\int_{t_k}^{t_{k+1}} \left(\frac{\partial^2 \tilde{u}}{\partial t^2} + \nu^2 \tilde{u}\right) \begin{Bmatrix} \cos(\nu \Delta t) \\ \sin(\nu \Delta t) \end{Bmatrix} dt = 0 \tag{106}$$

Let Δu be the difference between the exact solution u and the approximation $\tilde{u}$

$$u = \tilde{u} + \Delta u$$

One obtains, by substituting this expression into Eq. (54)

$$(d^2 \Delta u)/dt^2 + \nu^2 \Delta u + R(t) = 0$$

with

$$R(t) = (\partial^2 \tilde{u})/\partial t^2 + \nu^2 \tilde{u}$$

Hence, provided that for $\Delta u = 0$, $\Delta u^{\cdot} = 0$ for $t = t_k$

$$\Delta u(t) = \nu^{-1}\left[-\cos(\nu\Delta t)\int_{t_k}^{t} k(\tau)\sin(\nu\Delta\tau)\,d\tau + \sin(\nu\Delta t)\int_{t_k}^{t} k(\tau)\cos(\nu\Delta\tau)\,d\tau\right]$$

$$\Delta u^{\cdot}(t) = +\sin(\nu\Delta t)\int_{t_k}^{t} k(\tau)\sin(\nu\Delta\tau)\,d\tau + \cos(\nu\Delta t)\int_{t_k}^{t} k(\tau)\cos(\nu\Delta\tau)\,d\tau]$$

$$(107)$$

Hence, with Eqs. (106)

$$\Delta u(t_{k+1}) = 0, \quad \Delta u^{\cdot}(t_{k+1}) = 0$$

One sees that with the choice Eq. (105) of the weight function the errors in u and $u^{\cdot}$ vanish at the grid point t_{k+1}, and by induction at all subsequent grids points.

The argument is slightly different for the space dependence. The starting point is the equation

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial y^2} = 0$$

We compare an exact solution and an approximation which have the same dependence upon time. The exact solution is given by

$$\phi(y,t) = u(y)\exp(\pm i\nu t) = \exp(i\nu y)\exp(\pm i\nu t)$$

$$\phi_y(y,t) = du/dy\,\exp(\pm i\nu t) = i\nu\exp(i\nu y)\exp(\pm i\nu t)$$

One has specifically

$$u_k = \exp(i\nu kh)$$

$$u_k' = h^{-1} i\nu\exp(i\nu kh) \qquad (108)$$

The factor h^{-1} in u_k' occurs because by our original definition, the prime denotes the derivative with respect to t/h. The function u satisfies the equation

$$d^2u/dt^2 + \nu^2 u = 0$$

The approximation is assumed to have the form

$$\tilde{u}(y)\exp(i\nu t)$$

Now it is assumed that $\tilde{u}(y)$ is given by piecewise third degree polynomials with continuity of the function and of the first derivative. Let

$$\tilde{u}_k = \tilde{u}(kh)$$

and

$$\tilde{u}_k' = h^{-1}\frac{\partial\tilde{u}}{\partial y}(kh)$$

With tentative weight functions $w_i(\Delta y)$ one finds as conditions from which $\tilde{u}_k$ and $\tilde{u}_k'$ are determined

$$\int_{y_k}^{y_{k+1}}\left(\frac{\partial^2\tilde{u}}{\partial y^2} + \nu^2 u(y)\right) w_i(\Delta y)\,dy = 0 \qquad (109)$$

Previously we proceeded as follows. Particular solutions are obtained by setting

$$\tilde{u}_k = C\exp(i\mu kh)$$

$$\tilde{u}_k' = i\bar{C}\exp(i\mu kh)$$

This leads to a homogeneous system for C and $\bar{C}$. The elements of the governing matrix depends upon μ. The vanishing of its determinant establishes a relation between μ and ν (no matter how the weight functions are chosen). We set

$$\Delta u(y) = u(y) - \tilde{u}(y)$$

One then obtains

$$\frac{d^2}{dy^2}\Delta u + \nu^2\Delta u + R = 0$$

$$R = \frac{\partial^2\tilde{u}}{\partial y^2} + \nu^2\tilde{u}$$

Then

$$\Delta u_{k+1} - \Delta u_k = \nu^{-1}\left[\cos(\nu\Delta y)\int_{y_k}^{y_{k+1}} R(y)\sin(\nu\Delta y)\,dy - \sin(\nu\Delta y)\int_{y_k}^{y_{k+1}} R(y)\cos(\nu\Delta y)\,dy\right]$$

$$\Delta u_{k+1}' - \Delta u_k' = -\sin(\nu\Delta y\int_{y_k}^{y_{k+1}} R(y)\sin(\nu\Delta y)dy - \cos(\nu\Delta y)\int_{y_k}^{y_{k+1}} R(y)\cos(\nu\Delta y)\,dy$$

(110)

Now we impose the condition that the errors in Δu and $\Delta u'$ vanish at the grid points in the y-direction the left sides of the last equation vanish. In order for the right side to vanish, it is necessary that the integrals vanish separately.

$$\int_{y_k}^{y_{k+1}} \left(\frac{\partial^2 \tilde{u}}{\partial y^2} + \nu^2 \tilde{u}\right) \sin(\nu \Delta y)\, dy = 0$$

$$\int_{y_k}^{y_{k+1}} \left(\frac{\partial^2 \tilde{u}}{\partial y^2} + \nu^2 \tilde{u}\right) \cos(\nu \Delta y)\, dy = 0$$

These conditions are identical with Eqs. (109) if one chooses

$$\begin{aligned} w_1 &= \sin(\nu \Delta y) \\ w_2 &= \cos(\nu \Delta y) \end{aligned} \qquad (111)$$

If these conditions are satisfied, one obtains because of Eq. (108)

$$u_k = \tilde{u}_k = \exp(i\nu kh)$$

$$u_k' = \tilde{u}_k' = h^{-1} i\nu \exp(i\nu kh)$$

and then it follows that $\tilde{u}(y)$ is periodic iwy with period $\frac{\nu h}{2\pi}$. It does not matter how $\tilde{u}(y)$ is related to $\tilde{u}_k$ and $\tilde{u}_k'$ provided, of course, that if

$$\tilde{u}(y) = f(\tilde{u}_k, \tilde{u}_k')$$

then

$$\tilde{u}(y+h) = f(\tilde{u}_{k+1}, \tilde{u}_{k+1}')$$

The initial data will, in reality, contain linear combinations of waves $\exp(i\nu y)$ with different values of ν. The weight functions $\sin(\nu\Delta y)$ and $\cos(\nu\Delta y)$ are perfect only for one of them; for all other wave lengths some error will occur. At best, it is possible to tune the weight functions to a certain wave length, then it will be nearly correct for adjacent frequencies. The weight function for very long waves are given by

$$\lim_{\nu \to 0} \cos(\nu \Delta y) = 1$$

and

$$\lim_{\nu \to 0} (\nu^{-1} \sin(\nu \Delta y) = \Delta y$$

Figure 28 is similar to Figures 12 and 13. It shows for third degree weight functions the wave speed for different wave lengths with this choice of the weight functions. For short waves (large values of μ or ν) the result is certainly not better than the weight functions on which Figures 12 and 13 are based. It might be preferable to attune the weight function to a value of ν somewhat larger than zero.

The Duhamel solution of the inhomogeneous problem used in Eqs. (107) can be regarded as a representation of the Green's function for the ordinary differential equation (54)

The Green's function for the wave equation can be obtained by a Fourier analysis with respect to the y and t directions. The circular frequency ν will then vary from 0 to infinity. The weight functions which one would obtain by applying the above rationale are $w_1 = 1$, $w_2 = \Delta t$, $w_3 = \Delta y$ and $w_4 = \Delta y\ \Delta t$. They give equations for ϕ, ϕ_y, ϕ_t and ϕ_{yt} at each grid point. These weight functions are attuned to the low freqüency components of the Fourier decompositon.

The Green's function for the Laplace equation is given (except for a constant) by $\log(\vec{r}-\vec{r}')$. One then obtains as the effect of a residual $R(r')$

$$\Delta \phi(\vec{r}) = \iint R(\vec{r}') \log|r-r'|\, dx' dy'$$

From this expression one can derive weight functions by the requirement that the long distance effect of the residual from each element be small. Let the element center be at $\vec{r}' = \vec{r}_o$ with coordinates $x = x_o$, $y = y_o$. Then,

$|\vec{r}' - \vec{r}_o|$ is small in comparison to $|\vec{r} - \vec{r}_o|$. One obtains, by a development of $\log(\vec{r}' - \vec{r})$ with respect to $(x' - x_o)$ and $(y' - y_o)$

$$\iint_{\text{element}} R(\vec{r}') \log|r - r'| dx'dy' = \log|r - r_o| \iint_{\text{element}} R(\vec{r}') dx'dy'$$
$$+ \frac{x - x_o}{(\vec{r} - \vec{r}_o)^2} \iint_{\text{element}} R(\vec{r}')(x' - x_o) dx'dy' + \frac{y - y_o}{|\vec{r} - \vec{r}_o|^2} \iint_{\text{element}} R(\vec{r}')(y' - y_o) dx'dy'$$
$$+ \text{higher order terms}$$

This suggests three weight functions for each element, namely, $w_1 = 1$, $w_2 = \Delta x$ and $w_3 = \Delta y$. (One remembers that for the Laplace equation and bicubic elements one expresses the solution in terms of ϕ, ϕ_x and ϕ_y for each grid point.) The number of weight functions obtained by the above reasoning is correct. There is, of course, a question whether one should use these weight functions instead of those which arise from an extremum formulation. The same approach can be used to obtain weight functions for the three dimensional Laplace equation.

In a corresponding manner one can discuss the choice of the weight functions for the three dimensional Helmholtz equation. There one has as effect of a residual (except for a constant)

$$\iiint R(x', y', z') \frac{\cos(\mu|\vec{r} - \vec{r}'|)}{|\vec{r} - \vec{r}'|} dx'dy'dz'$$

The weight functions are obtained from an approximate representation of the Green's function

$$\frac{\cos(\mu|\vec{r} - \vec{r}'|)}{|\vec{r} - \vec{r}'|} = \frac{\cos(\mu|\vec{r} - r_o|)}{|\vec{r} - \vec{r}_o|} - \frac{\cos(\mu|\vec{r} - \vec{r}_o|)}{|\vec{r} - \vec{r}_o|^3}[(x - x_o)(x' - x_o) + (y - y_o)(y' - y_o)]$$
$$- \frac{\mu \sin(\mu|\vec{r} - \vec{r}_o|)}{(\vec{r} - \vec{r}_o)^2}[(x - x_o)(x' - x_o) + (y - y_o)(y' - y_o)]$$
$$+ \text{higher order terms}$$

The expression appearing in the second line dies out only as $1/(\vec{r}-\vec{r}\,')$, while that in the first line dies out as $1/(\vec{r}-\vec{r}\,')^2$. (Either $x - x_0$ or $y - y_0$ is of the same order as r.) The above form suggests weight functions $w_1 = 1$, $w_2 = \Delta x$, $w_3 = \Delta y$, $w_4 = \Delta z$, but strictly speaking the goal of suppressing long range effects is achieved only if $\mu(x'-x_0)$, $\mu(y'-y_0)$ and $\mu(z'-z_0')$ are small within the element under consideration. The size of admissible finite volumes must be smaller than the wave length.

The idea of making long range effects small fails for hyperbolic equations. The effect of a residual is given by

$$\iiint k(t',y',z')\,[(t-t')^2-(y-y')^2-(z-z')^2]^{-1/2}\,dt'\,dy'\,dz'$$

The square root vanishes at the characteristic cone through the point t', y', z', and for this vicinity a development of this expression is no longer feasible. It is impossible to supress long range effects. Some further insight is obtained if one considers the finite element concept in conjunction with the idea of characteristics.

SECTION IX

FINITE ELEMENT AND CHARACTERISTIC COORDINATES

The presence of characteristics imposes a definite structure to hyperbolic equations; one expects that this fact has some bearing on the implementation of the idea of weighted residuals. Perturbations (for instance the effects of a residual) propagate along characteristics. In Appendix VI the Green's function belonging to a two dimensional problem has been studied. A local source in a flow with a Mach number $\sqrt{2}$ gives a perturbation in the velocity only along the characteristics emanating from the point at which the source is located. Along these characteristics one has a step in the potential. The perturbation velocity has the direction normal to them; it is given by a δ function. Also the perturbation in the mass flow vector is given by a delta function; it has the direction of the characteristics. The effect of such a source does not die out with distance. In choosing weight functions for elliptic equations, one can use a plausibility argument. The long range effect of truncation errors can be reduced by postulating that within small regions the residuals counteract each other. This argument cannot be applied here: the effect of two sources which lie on different characteristics will not die out with distance (although it is confined to a narrow region). We study here to which extent this state of affairs can be taken into account by the choice of the element shape and of the weight functions.

Consider plane and axisymmetric flows at a Mach number $\sqrt{2}$. Then one has, respectively

$$-\phi_{xx} + \phi_{yy} = 0 \tag{112}$$

$$-\phi_{xx} + \phi_{rr} + \frac{1}{r}\phi_r \tag{113}$$

If small values of r are excluded, then it is practical to introduce in the axisymmetric case

$$\tilde{\phi} = r^{1/2}\phi \tag{114}$$

Then one obtains

$$-\frac{\partial^2\tilde{\phi}}{\partial x^2} + \frac{\partial^2\tilde{\phi}}{\partial r^2} + \frac{1}{4r^2}\tilde{\phi} = 0 \tag{115}$$

Introducing characteristic coordinates

$$\begin{aligned} \xi &= x + y \\ \eta &= x - y \end{aligned}$$

or (116)

$$\begin{aligned} \xi &= x + r \\ \eta &= x - r \end{aligned}$$

one obtains

$$\frac{\partial^2\phi}{\partial\xi\partial\eta} = 0 \quad \text{in the two dimensional case,} \tag{117}$$

and

$$\frac{\partial^2\phi}{\partial\xi\partial\eta} = \frac{\phi_\xi - \phi_\eta}{2(\xi-\eta)}$$

or (118)

$$r^{-1/2}\frac{\partial^2\tilde{\phi}}{\partial\xi\partial\eta} = r^{-1/2}\frac{\tilde{\phi}}{4(\xi-\eta)^2}$$

in the axisymmetric case (119)

The axisymmetric case has been included because simplicity of the plane problem might lead to faulty generalizations.

Now we consider the equation

$$\phi_{\xi,\eta} = f(\xi,\eta) \tag{120}$$

where $f(\xi,\eta)$ is either considered as known or given as a function of ϕ and its lower derivatives.

Assume that the element boundaries are characteristic lines. This means that we deal with a rectangular grid in the ξ,η plane. The solution in one quadrangel element is completely determined if one knows the values of ϕ along

two adjacent sides. From the characteristic conditions one then obtains the values of ϕ_η and ϕ_ξ along the other sides (ϕ_η along the side for which ξ = const, ϕ_ξ along the side for which η = const). The potential is obtained by integrations. The effect of the residual can be judged by considering the falsification of ϕ (or of ϕ_η and ϕ_ξ) along the newly computed sides of the quadrangle. This criterion is preferable to an evaluation of the residual at infinity because the errors do not die out. If the function $f(\xi,\eta)$ is known, then one has by a direct integration,

$$\phi_\eta(\xi=\xi_1,\eta) - \phi_\eta(\xi=\xi_0,\eta) - \int_{\xi_0}^{\xi_1} f(\tilde{\xi},\eta)\,d\tilde{\xi} = 0$$

$$\phi_\xi(\xi,\eta=\eta_1) - \phi_\xi(\xi,\eta=\eta_0) - \int_{\eta_0}^{\eta_1} f(\xi,\tilde{\eta})\,d\tilde{\eta} = 0$$

The values of $\phi_\eta(\xi = \xi_1,\eta)$ and $\phi_\xi(\xi,\eta = \eta_1)$ (and the values of ϕ which arise from them by an integration) is the only information needed in order to continue the computation in adjacent elements. Actually, one must approximate ϕ by means of a finite number of parameters. Equations for these parameters are obtained by applying weight functions. But since only the data along the element boundaries $\xi = \xi_1$ and $\eta = \eta_1$ are needed it is sufficient to use weight functions $w_i(\eta)$ along the segment $\xi = \xi_1$ = const and $w_j(\xi)$ along the segment $\eta = \eta_1$ = const. One therefore obtains

$$\int_{\eta_0}^{\eta_1} w_j(\tilde{\eta})\left\{\phi_\eta(\xi=\xi_1,\tilde{\eta}) - \phi_\eta(\xi=\xi_0,\tilde{\eta}) - \int_{\xi_0}^{\xi_1} f(\tilde{\xi},\tilde{\eta})\,d\tilde{\xi}\right\}d\tilde{\eta} = 0$$

and similarly for the second equation.
Hence, in an obvious manner

$$\int_{\eta_0}^{\eta_1}\int_{\xi_0}^{\xi_1} w_j(\tilde{\eta})\left[\phi_{\xi\eta}(\tilde{\xi},\tilde{\eta}) - f(\tilde{\xi},\tilde{\eta})\right]d\tilde{\xi}\,d\tilde{\eta} = 0 \qquad (121)$$

This is a weighted residual expression (The integration is extended over the whole element). The weight w_j depends only upon η. In the other case one obtains weights which depend only upon ξ.

In the axisymmetric cases we derived two forms of the differential equation depending upon which form one chosses one has as weight functions either

$$w_i(\xi) \quad \text{or} \quad w_i(\xi)\, r^{1/2}$$

as

$$w_j(\eta) \quad \text{or} \quad w_j(\eta)\, r^{1/2}$$

It is assumed that the residual is formed for the original differential equation (113) in each case. There is still some arbitrariness in the choice of the weights. For large values of r the difference is unessential. One has very little variation in one of the characteristic directions while in the other direction the choice of the weight function depends upon what kind of details one is willing to disregard.

In principle, the errors so admitted do not die out with distance. The reason for disregarding them is that they express unessential details. A finite element representation depends, as always, on a certain smoothness of the function that is to be represented. Discontinuities of the derivatives are admissible along the characteristics. (They might be introduced by discontinuities in the boundary conditions.) Such discontinuities at the characteristic element boundaries are compatible with the present formulation and if they are present, a finite element procedure based on the elements bounded by characteristics may give better results than the finite element procedures discussed in the preceding sections. To see which steps one has to take we consider the differential equation $\phi_{\xi\eta} + f(\xi,\eta) = 0$ and assume that in a characteristic quadrangle the function ϕ is represented by bicubics. This

requires that one knows at each of the four corners ϕ, ϕ_ξ, ϕ_η and $\phi_{\xi\eta}$. $\phi_{\xi\eta}$ is given by the differential equations. Known are the data along the lines 1, 2, and 1, 3 (See Figure 29).

Specifically, ϕ_1, $\phi_{\xi,1}$, $\phi_{\eta,1}$, ϕ_2, $\phi_{\xi,2}$, ϕ_3, $\phi_{\eta,3}$

unknown are $\phi_{\eta,2}$, $\phi_{\xi,3}$, $\phi_{\xi,4}$, $\phi_{\eta,4}$, and ϕ_4

Accordingly, one can use five weight functions. One of them is obviously the constant, which depends neither on ξ or η, for the other weight functions one will choose two which depend solely upon ξ, and two which depend solely upon η. Further details are omitted. Some modifications are required along a line for which initial conditions are given. This line must coincide with a characteristic, and one deals with triangular elements.

The stability analysis in all previous discussion has been carried out for the wave equation. In the present case one then considers particular solutions of

$$\phi_{\xi\eta} = 0$$

which are given by

$$\phi = f_1(\xi) + f_2(\eta)$$

For bicubic shape functions one has as representation in one element

$$\phi = \Sigma\, g_i(\xi) h_k(\eta)$$

where the g_i's and h_k's are polynomials of the third degree. Among expression of this form, there are some of the form $f_1(\xi)$ and $f_2(\eta)$. This means that the differential equation will be exactly satisfied. It follows that the method is stable in the same sense as one speaks of stability for the methods examined in preceding sections. Here we have neither damping nor dispersive errors; the only inaccuracies arise in satisfying the boundary conditions.

Of course the characteristics are not fixed a priori in nonlinear problems. Accordingly, one loses control over the shape of the elements. This will probably make programming more cumbersome.

An analogous approach to the three-dimensional problem leads to serious difficulties. In three dimensions one has characteristic surfaces which can be chosen in a great variety of ways. To generate such surfaces, one can, for instance, choose in an initial value plane, a two-dimensional mesh consisting of quadrangles and then construct the characteristic surfaces which emanate from these mesh lines. One obtains two surfaces for each line. If the lines are straight, one would obtain two characteristic planes for each line. By forming the lines of intersection of these planes with a noncharacteristic plane nearly perpendicular to them, one obtains a pattern similar to those for the two-dimensional characteristic method. But one has two families of lines in the initial value plane and thus, one obtains four different families of characteristic surfaces. The finite volumes bounded by such surfaces are not well suited for a computation. Moreover, these surfaces depend upon the choice of the original grid and a great number of choices are possible.

The characteristics in two dimensions and in three dimensions are lines or surfaces along which discontinuities may propagate. (Such discontinuities are introduced either by the initial or by the boundary conditions.) In the two-dimensional problem, characteristics along which singularities (or also steep gradients) propagate are readily found, they are members of the family of characteristics by means of which the computations are carried out. In the three-dimensional case it would be necessary to orient characteristic surfaces according to the discontinuities (and

steep gradients) as they are introduced by the initial or boundary conditions. This may make it necessary to change the definition of the elements as one goes along.

In a typical three-dimensional finite volume approach one might consider it as desirable to build up the whole flow field from finite volume elements which are bounded by characteristic surfaces in a manner which takes the regions of dependence into account. This is very difficult, if not impossible. Assume for instance that one has a rectangular grid in a noncharacteristic initial plane. The characteristic surfaces described above form four-sided pyramids which lie wholly in the region of influence of the data assigned within an initial quadrangle. However, one needs very complicated elements in order to fill during the next step the spaces between these pyramids.

A finite difference formulation based on the idea of characteristic surfaces is better able to cope with this problem because it defines the approximation only at the grid points.

In the two-dimensional problem a finite element procedure combined with the method of characteristic can also be motivated by considering the Green's function (see Section IX). The Green's function of the three-dimensional problem has a very complicated singularity along the characteristic cone. This makes it difficult, if not impossible, to derive from it , forms of the weight functions which are well fitted to the problem.

The singularity becomes more managable (in fact, it becomes rather close to the Green's functions for the two-dimensional problem), if along certain lines, the residuals are smooth functions and one carries out an integration over these lines.

One feasible method of approach can be described as follows. One introduces in an initial plane one family of nonintersecting curves. In a plane problem these would be straight lines in the direction in which ϕ and the velocities do not vary. In an axisymmetric problem they would be circles in a plane normal to the axis of symmetry. Then one forms characteristic surfaces which emanate from these lines. To be specific, at each point of one of these lines one constructs the characteristic cone. The characteristic surfaces mentioned above are the envelopes of these cones. The lines along which the characteristic surfaces are tangent to the cones are called bi-characteristics. One thus obtains elongated subvolumes with triangular or quadrangular cross sections in the initial elements, and quadrangular cross sections in subsequent elements. These cross sections lie approximately in planes determined by the bi-characteristics. In the plane case, these subvolumes are rods with appropriate triangular or quadrangular cross sections, in the axisymmetric case they are rings. The characteristic conditions contain only derivatives within a characteristic surface, one of the directions in which derivatives are formed is that of the bi-characteristic, the other one in the lengthwise direction of the rods or rings mentioned above.

The elements to be used in a finite element approach are cut out from these subvolumes by planes approximately determined by bi-characteristics. In the plane and axisymmetric case this leads directly to the familiar characteristic conditions. More general problems can probably be treated too, provided that the characteristic surfaces are oriented so that they coincide with surfaces where singularities of strong gradients occur. If this condition is not satisfied, then singularities or strong gradients will probably express themselves in the same manner as in the

usual finite difference methods, namely by oscillations around the correct values. If strong gradients are detected, then it is probably best to make a transition to a different, more suitably oriented set of characteristic surfaces.

The author has not been able to study the practical aspects of such a procedure.

SECTION X

GENERAL OBSERVATIONS AND SUMMARY OF SPECIFIC RESULTS

It was already mentioned in the introduction that the attempt to derive a finite element method for hyperbolic problems from a variational formulation is not likely to be useful; at best the idea is unnecessarily confining. In this report the method of weighted residuals is advocated instead.

We add that the idea of gaining an extremum formulation by postulating that the sum of the squares of the residual be minimized is of doubtful value. Interpreting this method in terms of weighted residuals, one finds that the weight is proportional to the residual. But the long distance effect of residuals is linear, no matter how large or small they are. Therefore, the weight should be independent of the magnitude of the residual. Minimizing the sum of the squares of the residuals suppresses short waves (because they give larger residuals at equal amplitude), but the method is fairly insensitive to long wave errors.

Hyperbolic problems differ from elliptic problems by the fact that the solutions do not automatically smooth out. In an elliptic problem the waviness of the solution at some distance from the boundary reflects the local waviness of the inhomogeneous term. In a hyperbolic problem the waviness is determined by the initial conditions the boundary conditions and the inhomogeneous part in all of the characteristic forecone pertaining to the point under consideration. Therefore, one cannot count on the smoothness of the solution.

One observes that solutions which are wavy in the space direction are also wavy in the time direction. Even short waves will not be damped. This fact is directly connected

with the existence of characteristic surfaces. Along such surfaces singularities may propagate; their attenuation with distance is only small. Assume that by the initial conditions a discontinuity (in the second derivatives, say) is introduced in the space direction. In order to represent such initial data by a Fourier analysis one needs all wave length including the very shortest ones. At a later time this singularity will still be present, although in a different position. If short waves were more strongly damped than long ones such singularities would vanish.

This explains why stability is particularly critical in hyperbolic problems. In essence, the individual particular solutions have neutral stability (neither damped nor excited). The error introduced by an approximation may change neutral stability into instability. In elliptic problems where the particular solutions die out with distance, the same kind of error will falsify the rate of attenuation, but it is unlikely that an exact solution which is stable will change into one that is unstable. The situation is particularly favorable because the contributions of those particular solutions which are strongly falsified are small to begin with and furthermore, because they are most strongly damped.

The finite difference as well as the finite element method presupposes that the solutions are smooth enough so that they can be approximated within the inidividual elements by simple standardized expressions. In elliptic, as in hyperbolic problems, one must choose a grid which is suitable to represent the essence of the desired solution but disregards short wave roughness. In elliptic equations the mesh size satisfying this requirement is mainly determined by the boundary data. In principle, one might use a larger grid at some distance from the boundaries. In hyperbolic problems the same waviness is to be expected throughout the field.

The discussions carried out in this report show the effect of truncation errors. Let us confine our attention to the semi-discretized methods investigated in Sections II through VII. If the wave speeds are reproduced with sufficient accuracy for the range of wave lengths which is important, then the truncation error remains within acceptable bounds. Shorter waves in the initial data and in the inhomogeneous part are falsified because of truncations errors, but it is assumed that these components are initially small. They will remain small unless the method is unstable for these wave lengths. For high accuracy machines rounding errors are usually very small. In addition, they have a random character. Accordingly, one will obtain acceptable solutions even though short waves are not damped. This optimistic assessment may not hold for nonlinear problems.

The primary reason that the Courant number for explicit finite difference methods applied to hyperbolic equations must be smaller than 1 is stability. However, this restriction is also needed from the point of view of accuracy, because the solutions have the same waviness in the space and in the time directions. If one admits Courant numbers greater than 1, then one loses information contained in the approximation to the initial conditions. There may, however, be problems where a mesh finer than required for reasons of accuracy is used in the space direction.

The latent presence of short waves makes itself felt in the semidiscretized approach during the integration of the resulting system of ordinary differential equations. In an explicit predictor corrector method the stability limit is about the reciprocal value of the largest eigenvalue. Usually this limits the step size even if the initial conditions do not contain particular solutions pertaining to the large eigenvalues which cause instability because such particular solutions are excited by the truncation errors of

the integration scheme. Implicit methods are sometimes regarded as a panacea, but this is not entirely correct. Basically, one deals with stiff differential equations. Familiar solvers for stiff differential equations are effective for problems in which the large eigenvalues which are responsible for the stiffness have large negative real parts while the imaginary part is fairly small. Under these conditions they allow the use of large integration intervals in regions where the solution is sufficiently smooth. Usually, they are not stable if the real part of the critical eigenvalues is small and the imaginary part is large. This is the present situation.

Our discussions have been restricted to rectangular elements and bilinear, biquadratic or bicubic shape functions. (This makes it possible to discuss space and time dependence independently.) The accuracy of a method is judged by the falsification of the wave speed (and, if necessary, also of the amplitude) for different wave lengths. Results of this kind are shown in a number of graphs. If one uses quadratic or cubic instead of linear shape functions, then one has for the approximation of the solution in the space direction twice as many parameters per grid point. Each discretization process suppresses waves in the space direction. By doubling the number of parameters per grid point one admits wave lengths down to one half of the original limiting wave length. In this sense, a representation by quadratic or cubic shape function is equivalent to one by linear shape functions with half of the grid size. The computational effort depends approximately upon the overall number of parameters. We have used the accuracy of the wave speeds as a criterion for a comparison of different methods. The mesh size for linear shape functions is chosen one half of that for quadratic or cubic shape functions; then linear and cubic elements include

bicubic shape functions. If one uses bilinear shape functions and weight functions of the same kind for space and time, then the truncation errors cancel if the Courant number is 1. The method is unstable if the Courant number exceeds one. (An exact integration of the differential equations for the time dependence corresponds to Courant number zero.) The favorable results obtained with Courant number one have their counterpart in finite difference methods.

Occasionally, particularly in the development of computational procedures for the transonic problem, it is emphasized that one must use a difference procedure which reflects the marching direction. A scheme of the kind just sketched does not have this property; a marching direction is defined by the way in which the initial conditions are prescribed.

Seen under the point of view of weighted residual procedure the method just described is somewhat unsatisfactory. In computing the potential at a new time station it considers not only the residual in the time interval that is newly computed, but also in the preceding interval over which one no longer has any control.

A method using linear shape functions has been investigated in which the weight function is constant through the interval for which the computation is carried out (say from t_k to t_{k+1} with an infinitesimal overlap into the region t_{k-1} to t_k. The overlap is needed in order to take into account the delta function in the second derivatives which arises at the point t_k because of a jump in the first derivatives which is unavoidable with linear shape functions. One obtains a method which is stable for all Courant numbers, the solutions for shorter waves are rather strongly damped and one is forced into rather small time intervals in order to obtain an acceptable accuracy.

It is impossible to use the same weight functions for the space and the time directions for third degree shape functions for the method becomes unstable at all Courant numbers. The reason is rather interesting. For third degree shape functions one obtains the solutions at time t_{k+1}; that is, the values of u_{k+1} and $\dot{u}_{k+1}$ from equations which contain the (known) values of u_k, $\dot{u}_k$, u_{k-1} and $\dot{u}_{k-1}$. In order to solve the differential equation in the interval between t_k and t_{k+1} it suffices if one known u_k and $\dot{u}_k$. Speaking in the language of numerical integration techniques for ordinary differential equations one would say that the present approach includes spurious solutions. These are particular solutions of the homogeneous discretized system which are not related to particular solutions of the original ordinary differential equation. In the present case there are two such spurious solutions and one of them is always undamped even if one keeps the Courant number below one. (For the other particular solutions one has again a cancellation of truncation errors for space and time if the Courant number is one.)

One can devise a procedure based on cubic shape functions in which no spurious solutions are encountered. As before, one makes the transition from point t_k to point t_{k+1} by third degree polynomials, but the weight functions are now confined to this interval. This eliminates the objection regarding the use of the residual in a preceding interval to determine the solution in the interval under investigation. We have studied this procedure using a weight function 1 over an initial part of the interval, namely, $t_k < t < t_k + c(t_{k+1} - t_k)$, $0 < c < 1$, and again over the remaining part $t_k + c(t_{k+1} - t_k) < t < t_{k+1}$. All particular solutions are unstable for $c < 1/2$. For $c = 1/2$ and a limited Courant number the method is stable, the eigenfunctions are undamped. There are, however, Courant numbers (somewhat exceeding one) for which unstable particular solutions exist. For $c > 1/2$ and

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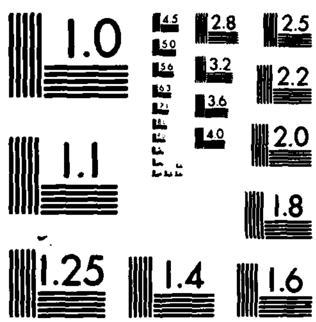

MICROCOPY RESOLUTION TEST CHART
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a Courant number limited approximately to a value below 1 the method is stable. The particular solutions are slightly damped. But for certain higher Courant numbers there are always unstable particular solutions. The procedure is stable for all Courant numbers in the limit $c = 1$. Then the weight is constant throughout the interval and in addition one has a delta function weight at the end of the interval. This amounts to a mixture of finite element method and collocation method. Even in this limit the falsification of long waves is not too large.

It was mentioned above that from the point of accuracy Courant numbers larger than one are without interest; at least if the shortest wave lengths which are included by the discretization in space direction are of significance. Cases where one will admit large Courant numbers occur in transonic problems because they include the vicinity of the sonic line. In Section VII a simple problem is discussed in which the Mach number assumes the value one at one boundary of the region. In the direction normal to the flow direction an evenly spaced mesh is used. For each length of the interval one obtains a maximum value of μ. One can therefore always choose an interval in the downstream direction small enough to guarantee stability. If the mesh size in the direction normal to the flow is multiplied by a factor α and one wants to maintain the same Courant number, then the mesh size in the downstream direction is multiplied by a factor $\alpha^{3/2}$. In this particular problem it is possible to find particular solutions by a product hypothesis. From the ordinary differential equation which arises for the direction normal to the flow, one obtains an eigenvalue problem as well for the ordinary differential equation as for its discretized form. For long waves, the eigenfunctions for the differential equation and for its discretized form are, of course, very similar. The short wave eigenfunctions are, however, entirely

unrelated. For the differential equation they are, in essence, given by Airy's functions. For the finite difference formulation the highest eigenfunctions are nearly zero (and oscillatory) at some distance from the boundary which corresponds here to Mach number 1. These particular solutions are significant only at the first few mesh points. As contributions to the solutions of the original partial differential equation problems these particular solutions are, of course, meaningless. If one fails to observe proper Courant number restrictions, then these particular solutions may increase as one moves downstream. The falsification so introduced will manifest itself in some roughness in the vicinity of the line corresponding to Mach number 1, but at some distance from it the error will be very small.

In the examples considered up to this point the weight functions had been chosen on intuitive grounds, either because of their resemblance to the shape functions or, for reasons of simplicity as constants. Section VIII considers a scalar differential equation with a fixed value of μ (in essence the reciprocal of the wave length). Then one can discuss the effect of a residual by writing down the solution of the inhomogeneous differential explicitly. We have considered such a solution in the form of the Duhamel's integral. This can be regarded as a solution in terms of the Green's function belonging to this particular one dimensional problem. The integrals occurring here have the form of weighted residual expressions. For this particular problem one obtains exact expressions at the grid points if the solutions of the homogeneous problems are taken as weight functions. In reality, one deals, of course, simultaneously with a whole spectrum of values of μ. At best, one can choose weight functions which are close to ideal within some range of values of μ. But the fact is of interest that one can attune the weight functions

to a certain frequency. In an example we have chosen $\mu = 0$ for this value. The weight functions are then given for each interval by a constant and by a linear function. From our numerical results we cannot claim that this method is superior to the method discussed earlier, but the author believes that the theoretical insight is valuable. A wider frequency could probably be covered if one does not attune the method to frequency zero. In the elliptic region one might start directly from the Green's function and impose the requirement that the lowest order terms in the development of the Green's functions with distance vanish. For the Laplace equation this gives linear weight functions. For the Helmholtz equation there are always some contributions of the lowest order terms which vanish with distance only weakly. They become small only if the mesh is sufficiently fine.

The same idea applied to hyperbolic problems fails, for the hyperbolic distance is zero for points that lie on the characteristics through the points where the residuals occur. The weight functions for hyperbolic problems used in the preceding analysis are suitable for long waves in time and space. Short waves are treated inaccurately, at best. A finite element of this kind is therefore unsuited to treat discontinuities which propagate along characteristics.

For two dimensional problems it is possible to combine the idea of finite elements with the concept of characteristics. This is done in Section IX. The shape functions are then defined in characteristic quadrangles. Important is the result that the weight functions should be constant in one of the characteristic directions. With this choice it becomes possible that contributions of residuals which lie on the same characteristic cancel each other. The desirability of this form of weight functions becomes obvious again if one considers the Green's function of this problem. A finite

element method carried out in this manner differs only slightly from the method of characteristics. Within an element discontinuities are suppressed, but along the element boundary discontinuities are admitted because they do not contribute to the residual.

The method of characteristics is not easily carried over to three dimensions and no conclusions in this direction have been reached.

The picture that emerges from these discussions is somewhat discouraging. A Courant number limitation always exists, if not for reasons of stability then for regions of accuracy. Higher (third) order elements are advantageous because they give for long wave lengths a much better representation of the wave speed than linear elements. The use of third order splines is probably advantageous because it suppresses short waves which have an undesirable effect on the time integration and still give a good representation for long waves. The method of characteristics possibly in combination with the finite element concept may still prove a superior alternative.

The report goes somewhat further than an evaluation of different methods by means of sample problems. It relates the results that could be found in this manner to the behavior of particular solutions of the approximating equations. In this manner certain inherent difficulties directly connected with the nature of hyperbolic equations become obvious. The approaches discussed here may not be the last word in this problem area. Other more ingenious approaches may well exist, but even they will encounter the same obstacles and then a familiarity with these phenomena may be valuable.

APPENDIX I

THE VARIATIONAL FORMULATION OF REDDY

A variational formulation for some hyperbolic problems which include the boundary conditions in a proper manner has been given by Reddy (Ref. 1). The author believes that a variational principle by itself (in contrast to an extremum principle) is not particularly useful for numerical work. The present appendix shows in a simple example the essence of Reddy's formulation. Consider the one-dimensional problem

$$u'' + \rho_0 u - f(t) = 0 \tag{A1}$$

with initial conditions of the type encountered in a hyperbolic problem

$$u(0) = u_0 \; ; \quad u'(0) = u'_0 \tag{A2}$$

The independent variable is t, the derivative of a function with respect to its argument is denoted by prime. Let the final value of t be $t_o > 0$.

Now consider the functional

$$\psi(u) = -u'(0)u(t_0) + \int_0^{t_0} [\tfrac{1}{2} u'(\tau)u'(t_0-\tau) + \tfrac{1}{2}\rho_0 u(\tau)u(t_0-\tau) - f(\tau)u(t_0-\tau)]\,d\tau \tag{A3}$$

We form the variation of $\psi(u)$, postulating that the functions u in competition satisfy eqs. (A2). Then one has

$$\delta u(0) = 0 \; ; \quad \delta u'(0) = 0 \tag{A4}$$

One obtains

$$\begin{aligned}\delta\psi(u) = -u'(0)\,\delta u(t_0) + \int_0^{t_0} \{ & \tfrac{1}{2}\delta u'(\tau)u'(t_0-\tau) + \tfrac{1}{2}u'(\tau)\delta u'(t_0-\tau) \\ & + \tfrac{1}{2}\rho_0\,\delta u(\tau)u(t_0-\tau) + \tfrac{1}{2}\rho_0 u(\tau)\delta u(t_0-\tau) \\ & - f(\tau)\,\delta u(t_0-\tau)\}\,d\tau\end{aligned}$$

One obtains by carrying out an integration by parts in the first two terms of the integrand

$$\delta\psi(u) = -u'(0)\,\delta u(t_0) + \tfrac{1}{2}\left[\delta u(\tau)u'(t_0-\tau) - u'(\tau)\,\delta u(t_0-\tau)\right]\Big|_{\tau=0}^{\tau=t_0}$$
$$+ \int_0^{t_0}\Big[\tfrac{1}{2}\delta u(\tau)u'(t_0-\tau) + \tfrac{1}{2}u''(\tau)\,\delta u(t_0-\tau)$$
$$+ \tfrac{1}{2}\rho_0\,\delta u(\tau)u_0(t_0-\tau) + \tfrac{1}{2}\rho_0\,u(\tau)\,\delta u(t_0-\tau) - f(\tau)\,\delta u(t_0-\tau)\Big]d\tau$$

In the first and third terms of the integrand the umbral variable τ is replaced by $t_0 - \tau$. In addition, we evaluate the terms outside of the integral

$$\delta\psi(u) = -u'(0)\,\delta u(t_0) + \tfrac{1}{2}\left[\delta u(t_0)u'(0) - \delta u(0)u'(t_0) - \delta u(0)u'(t_0) + \delta u(t_0)u'(0)\right]$$
$$+ \int_0^{t_0}\left[u''(\tau) + \rho_0 u(\tau) - f(\tau)\right]\delta u(t_0-\tau)\,d\tau \tag{A5}$$

The terms outside of the integral vanish because of the boundary conditions Eqs. (A2) and (A4). By the requirement that the variation of $\delta\psi(u)$ vanish one obtains indeed Eq. (A1).

The last expression can be interpreted as a weighted residual formulation. The weight functions are $\eta(t) = \delta u(t_0-t)$. Let the shape functions be restricted to some subspace of the space of admissible functions. Strictly speaking one cannot maintain that the shape functions $\eta(t)$ belong to the same function space, for the shape functions and their first derivatives vanish for argument 0, the shape functions and their derivative vanish for argument t_0. The property characteristic of the usual variational formulations, namely that weight functions and shape functions are taken from the same subspace no longer applies. The method would probably encounter difficulties if one uses an uneven grid in t. One notices furthermore that the procedure requires that $\rho(t) = \rho(t_0-t)$ (in our example we have assumed ρ_0 = const). These are obstacles to the application of Reddy's formulation to numerical work.

APPENDIX II

OBSERVATIONS REGARDING THE TREATMENT OF BOUNDARY CONDITIONS IN THE METHOD OF WEIGHTED RESIDUALS

The observations made in this appendix arose from discussions which I had with Dr. Soliman about the method of weighted residuals. Although they do not refer specifically to hyperbolic problems they are included here because they serve to round out the knowledge about the method of weighted residuals. The essential idea is already seen in a one-dimensional problem. Consider the differential equation

$$-\phi''(x) + g(x)\phi(x) + f(x) = 0 \tag{A6}$$

with boundary conditions

$$\phi(0) = 0 \tag{A7}$$

$$c_1\phi(L) + c_2\phi'(L) + c_3 = 0 \tag{A8}$$

Including the second boundary condition in a weighted residual formulation we write

$$\int_0^L [-\phi''(x) + g(x)\phi(x) + f(x)]\psi(x)dx + C[c_1\phi(L) + c_2\phi'(L) + c_3] = 0 \tag{A9}$$

where ψ is a weight function and C is a single weight. The functions ϕ in competition are assumed to satisfy the boundary condition

$$\phi(0) = 0$$

We postulate

$$\psi(0) = 0$$

The requirement of existence of second derivatives in ϕ can be relinquished if one carries out an integration by parts. One obtains, after substitution of the boundary conditions

$$\int_0^L (\phi'(x)\psi'(x) + g(x)\phi(x)\psi(x) + f(x)\psi(x))dx - \phi'(L)\psi(L) + C(c_1\phi(L) + c_2\phi'(L) + c_3) = 0 \tag{A10}$$

If one chooses C independent of the weight function ψ, then one obtains directly the condition Eq. (A8); the admissible functions ϕ must satisfy the boundary conditions at $x = L$ as well as $x = 0$.

According to Soliman, one can simplify the problem by choosing

$$C c_2 = \psi(L) \tag{A11}$$

with this choice $\phi'(L)$ vanished in Eq. (A10). This may well be worthwhile particularly in multidimensional problems because it obviates the need to evaluate the derivative in the direction of the conormal to the boundary.

To justify this special choice we rewrite Eq. (A9) expressing C by Eq. (A11). Then one has

$$\int_0^L \left\{ -\phi''(x) + g(x)\phi(x) + f(x) + \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \delta(x, L-\varepsilon)\left[\frac{c_1}{c_2}\phi(x) + \phi'(x) + \frac{c_3}{c_2} \right] \right\} \psi(x)\,dx = 0 \tag{A12}$$

Here the term generated by the boundary condition is included in the integral by means of the delta function $\delta(x, L-\varepsilon)$. The δ singularity approaches the point $x = L$ from the inside of the interval as $\varepsilon \to 0$. One sees that the failure to satisfy the boundary condition at $x = L$ appears in the weighted integral for the residual in the form of a delta function. There are other points within the interval where the residual is allowed to have delta singularities, these are points where ϕ' has jumps. The special choice of C in Eq. (A11) therefore has the effect of balancing the failure to satisfy the boundary conditions against other residuals caused in the differential equation by the approximation to ϕ. This is, of course legitimate. The situation is similar for multidimensional problems. The boundary terms in Eq. (A12) are then replaced by integrals over the boundary surface.

The factor by which the delta function is multiplied tends to infinity if c_2 tends to zero. Then this approach must be abandoned. In this case the functions ϕ must satisfy the boundary conditions at $x = L$ exactly.

In the problem at hand one has also a variational formulation, namely

$$\delta\left\{\int_0^L \left[\frac{1}{2}\phi'(x)^2 + \frac{1}{2}\rho_0(x)\phi^2(x) + f(x)\phi(x)\right]dx + \frac{1}{2}\frac{c_1}{c_2}\phi^2(L) + \frac{c_3}{c_2}\phi(L)\right\} = 0$$

In the case $c_2 = 0$, that is, if the value of ϕ is prescribed at the boundary, one can proceed in different ways. In one formulation one postulates

$$\delta\int_0^L \left[\frac{1}{2}\phi'(x)^2 + \frac{1}{2}\rho_0(x)\phi^2(x) + f(x)\phi(x)\right]dx = 0 \tag{A13}$$

and postulates that the admissible functions ϕ satisfy the boundary conditions at $x = 0$ and $x = L$. Alternatively, one can introduce the boundary conditions at $x = L$ by means of a Lagrange multiplier and disregard the boundary condition at $x = L$ during the variations. This formulation is discussed because it has some similarity to Eq. (A10). However, we shall see that the two formulations are different in principle. One has

$$\delta\left[\int_0^L \left[\frac{1}{2}\phi'(x)^2 + \frac{1}{2}\rho(x)\phi^2(x) + f(x)\phi(x)\right]dx + \lambda\left[\phi(L) - \frac{c_3}{c_1}\right]\right] = 0 \tag{A14}$$

Hence

$$\int_0^L \left(-\phi''(x) + \rho(x)\phi(x) + f(x)\right)\delta\phi(x)\,dx + \left(\phi'(L) + \lambda\right)\delta\phi(L) = 0$$

and

$$\begin{aligned} -\phi''(x) + \rho(x)\phi(x) + f(x) &= 0 \\ \phi'(L) + \lambda &= 0 \end{aligned} \tag{A15}$$

The variation vanishes if one solves the original differential equation with the boundary conditions $\phi(0) = 0$ and $\phi'(L) = -\lambda$.

This can be done for any value of λ. The desired solution is obtained if one determines λ so that the original boundary condition

$$\phi(L) = -\frac{C_3}{C_1}$$

is satisfied.

In a weighted residual approach one would write

$$\int_0^L (-\phi''(x) + g(x)\phi(x) + f(x))\,\psi(x)\,dx = 0$$

and

$$C(\phi(L) - C_3/C_1) = 0 \qquad \text{(A16)}$$

Eqs. (A16) and (A15) are not identical. In the approach with Langrange multipliers one computed in principle a family of solutions which satisfy a different boundary condition specified by the choice of λ. From this family one then selects the one which satisfies the boundary condition actually prescribed. In a weighted residual approach, the boundary conditions are introduced directly.

Functions ϕ which fail to satisfy the boundary conditions give approximations with a jump of ϕ at the boundary. Such a jump could be accommodated by means of the derivative of a δ function. But such derivatives cannot be dealt with by a method of weighted residuals. This is important in multi-dimensional problems where it is frequently impossible to find shape functions ϕ which satisfy the boundary conditions exactly. It is preferable to consider the error introduced by the failure to satisfy the boundary conditions, by itself, rather than to attempt to include it in an overall weighted residual formulation.

APPENDIX III

CHECK EXPRESSIONS FOR SOME FORMULAE OF SECTION II

The Eqs. (11) and (13) of cases 2a and 2b respectively can be checked by substituting the values of ϕ_{k-1}, ϕ_k and ϕ_{k+1} for three piecewise linear functions. For such functions the operator on the left of these equations must give the expressions which one obtains by substituting them and the chosen weight functions into Eq. (7).

The following expressions will be chosen

$$\phi(y,t) = C(t) \tag{A17a}$$

$$\phi(y,t) = \frac{\Delta y}{h} C(t) \tag{A17b}$$

$$\phi(y,t) = \begin{cases} 0, & \Delta y < 0 \\ \Delta y/h\, C(t) & \Delta y > 0 \end{cases} \tag{A17c}$$

In Case 2a the weight functions are given by Eq. (10). One obtains by substituting Eqs. (10) and (A17a) into Eq. (7)

$$\frac{d^2C}{dt^2}\int_{-\infty}^{+\infty} w(\Delta y/h)\, d(\Delta y/h) - \int_{-\infty}^{+\infty} 0\; w(\Delta y/h)\, d(\Delta y/h) = \frac{d^2C}{dt^2} \tag{A18}$$

The expression (A17b) substituted into Eq. (7) gives zero because $\frac{\partial^2\phi}{\partial t^2}$ and $\frac{\partial^2\phi}{\partial y^2}$ are antisymmetric and $w(\Delta y)$ is symmetric.

The expression (A17c) substituted into the first term of Eq. (7) gives

$$\frac{d^2C}{dt^2}\int_0^1 (\Delta y/h)\, N_1(\Delta y/h)\, d(\Delta y/h) = \frac{d^2C}{dt^2}\int_0^1 (\Delta y/h)(1-(\Delta y/h))\, d(\Delta y/h) = \frac{1}{6}\frac{d^2C}{dt^2}$$

The second term must be evaluated in the sense of generalized functions

$$C(t)\int_{-\infty}^{+\infty} \frac{d^2\phi}{dy^2} w(\Delta y/h)\, d(\Delta y/h) = C(t)h^{-2}\int^1 \frac{d^2\phi}{d(\Delta y/h)^2} w(\Delta y/h)\, d(\Delta y/h)$$

$$= C(t)h^{-2}\left[\frac{d\phi}{d(\Delta y/h)} w(\Delta y/h)\Big|_{-\varepsilon}^{1} - \int_0^{0-\varepsilon} \frac{\partial\phi}{\partial(\Delta y/h)}\frac{dw}{d(\Delta y/h)} d(\Delta y/h)\right] = C(t)h^{-2}\int_0^1 d(\Delta y/h) = C(t)h^{-2}$$

One obtains, therefore, for case (2b) with the "test" function Eq. (A17c)

$$\frac{1}{6}\frac{d^2C}{dt^2} - h^{-2}C(t) \tag{A19}$$

One finds from (A17c)

$$\phi_{k-1} = 0, \quad \phi_k = 0, \quad \phi_{k+1} = C(t) \tag{A20}$$

One obtains the expression (A19) if one substitutes Eq. (A20) into Eq. (11).

Case 2b.

With the weight given by Eq. (12) one obtains in the same manner

d^2C/dt^2	for Eqs. (A.17a) and (12)
0	for Eqs. (A.17b) and (12)
$d^2C/dt^2 - h^{-2}C$	for Eqs. (A.17c) and (12)

Cases 3.

In the resulting formulae, Eqs. (16), (18) and (20) there appears six unknowns, $\phi_{k-1}(t)$, $\phi'_{k-1}(t)$, $\phi_k(t)$, $\phi'_k(t)$, $\phi_{k+1}(t)$ and $\phi'_{k+1}(t)$. For a check of these equation one needs six independent functions ϕ, which are piecewise of third degree and continuous in function and first derivative. One notices that the first and second equations of the equation pairs (16), (18) and (20) are automatically satisfied if ϕ is antisymmetric and symmetric, respectively with respect to the point $y = y_k$.

A convenient choice of the test functions is

$$\phi = C(t) \tag{A21a}$$

$$\phi = C(t)\ \Delta y/h \tag{A21b}$$

$$\phi = C(t)(\Delta y/h)^2 \tag{A21c}$$

$$\phi = C(t)(\Delta y/h)^3 \tag{A21d}$$

$$\phi = \begin{cases} -C(t)(\Delta y/h)^3 & \Delta y/h < 0 \\ C(t)(\Delta y/h)^3 & \Delta y/h > 0 \end{cases} \tag{A21e}$$

$$\phi = \begin{cases} -C(t)(\Delta y/h)^2 & \Delta y/h < 0 \\ C(t)(\Delta y/h)^2 & \Delta y/h > 0 \end{cases} \tag{A21f}$$

Case 3a.

Substituting the expression (A21) together with the weight functions (15) into the operator on the left of Eq. (7), one obtains

d^2C/dt^2	for Eqs. (A21a) and (15a)
0	for Eqs. (A21a) and (15b)
0	for Eqs. (A21b) and (15a)
$1/15\ d^2C/dt^2$	for Eqs. (A21b) and (15b)
$2/15\ d^2/dt^2 - 2h^{-2}C$	for Eqs. (A21c) and (15a)
0	for Eqs. (A21c) and (15b)
0	for Eqs. (A21d) and (15a)
$\frac{2}{105}\frac{d^2C}{dt^2} - \frac{-2}{5}h^{-2}C$	for Eqs. (A21d) and (15b)
$\frac{1}{14}\frac{d^2C}{dt^2} - \frac{9}{5}h^{-2}C$	for Eqs. (A21e) and (15a)
0	for Eqs. (A21e) and (15b)
0	for Eqs. (A21f) and (15a)
$\frac{1}{30}\frac{d^2C}{dt^2} - \frac{1}{3}h^{-2}C$	for Eqs. (A21f) and (15b)

These results can be used to check Eqs. (16)

Case 3b.

The weight functions are given by Eqs. (17), the test functions by Eqs. (A21). The following results are obtained by substitution into Eq. (7)

d^2C/dt^2	for Eqs. (A21a) and (17a)
0	for Eqs. (A21a) and (17b)
0	for Eqs. (A21b) and (17a)
$\frac{1}{4}\, d^2C/dt^2$	for Eqs. (A21b) and (17b)
$\frac{1}{12}\, d^2C/dt^2 - 2h^{-2}C$	for Eqs. (A21c) and (17a)
0	for Eqs. (A21c) and (17b)
0	for Eqs. (A21d) and (17a)
$\frac{1}{32}\, d^2C/dt^2 - \frac{3}{2}\, h^{-2}C$	for Eqs. (A21d) and (17b)
$\frac{1}{32}\, d^2C/dt^2 - \frac{3}{2}\, h^{-2}C$	for Eqs. (A21e) and (17a)
0	for Eqs. (A21e) and (17b)
0	for Eqs. (A21f) and (17a)
$\frac{1}{12}\, d^2C/dt^2 - 2h^{-2}C$	for Eqs. (A21f) and (17b)

These results can be used to check Eqs. (18).

Case 3c.

The weight functions are given by Eqs. (19). The test functions are given by Eq. (A21). The following results are obtained by substitution into Eq. 7.

$\frac{1}{2}\, d^2C/dt^2$	for Eqs. (A21a) and (19a)
$\frac{1}{2}\, d^2C/dt^2$	for Eqs. (A21a) and (19b)

0 for Eqs. (A21b) and (19a)

$\frac{1}{4}\, d^2C/dt^2$ for Eqs. (A21b) and (19b)

$\frac{1}{96}\, d^2C/dt^2 - h^{-2}C$ for Eqs. (A21c) and (19a)

$\frac{13}{96}\, d^2C/dt^2 - h^{-2}C$ for Eqs. (A21c) and (19b)

0 for Eqs. (A21d) and (19a)

$\frac{5}{64}\, d^2C/dt^2 - \frac{3}{2}\, h^{-2}C$ for Eqs. (A21d) and (19b)

$\frac{1}{512}\, d^2C/dt^2 - (3/8)h^{-2}C$ for Eqs. (A21e) and (19a)

$\frac{5}{64}\, d^2C/dt^2 - \frac{3}{2}\, h^{-2}C$ for Eqs. (A21e) and (19b)

0 for Eqs. (A21f) and (19a)

$\frac{13}{96}\, d^2C/dt^2 - h^{-2}C$ for Eqs. (A21f) and (19b)

These results are used to check Eq. 20.

Cases 4.

The resulting formulae, Eqs. (21) and (23) are correct for continuous piecewise quadratic functions. In Eqs. (22) and (24) there appear five different functions. Therefore, one needs five test functions. The following expressions are suitable.

$$\phi = C(t) \tag{A22a}$$

$$\phi = C(t) \quad \Delta y/h \tag{A22b}$$

$$\phi = C(t)\,(\Delta y/h)^2 \tag{A22c}$$

$$\phi = \begin{cases} -C(t)\ \ \Delta y/h & -1 < \Delta y/h \le 0 \\ C(t)\ \ \Delta y/h & 0 < \Delta y/h \le 1 \end{cases} \tag{A22d}$$

$$\phi = \begin{cases} -C(t)(\Delta y/h)^2 & -1 < \Delta y/h \leq 1 \\ C(t)(\Delta y/h)^2 & 0 < \Delta y/h \leq 1 \end{cases} \qquad \text{(A22e)}$$

There is no need to examine the test function Eq. (A22e) separately. Weight functions (21a) and (23a) gives zero for reasons of symmetry, for weight functions Eqs. (21b) and (23b) it coincides with case (A22c) because the weights are identically equal to zero outside of the region $0 \leq \Delta y/h \leq 1$. The following results are obtained by substitution into Eq. (7).

Case 4a.

The weight functions are given by Eqs. (21) the test functions by Eqs. (A22). The following results are obtained by substitution into Eq. (7).

d^2C/dt^2	for Eqs. (A22a) and (21a)
$\frac{2}{3}\, d^2C/dt^2$	for Eqs. (A22a) and (21b)
0	for Eqs. (A22b) and (21a)
$\frac{1}{3}\, d^2C/dt^2$	for Eqs. (A22b) and (21b)
$\frac{1}{6}\, d^2C/dt^2 - 2h^{-2}C$	for Eqs. (A22c) and (21a)
$\frac{1}{5}\, d^2C/dt^2 - \frac{4}{3}\, h^{-2}C$	for Eqs. (A22c) and (21b)
$\frac{1}{3}\, d^2C/dt^2 - 2C$	for Eqs. (A22d) and (21a)
$\frac{1}{3}\, d^2C/dt^2$	for Eqs. (A22d) and (21b)

The results are used to check Eq. (22).

Case 4b.

The weight functions are given by Eqs. (23) and the test functions by Eqs. (A22). The following results are obtained by substitution into Eq. (7).

$\frac{1}{2}\, d^2C/dt^2$ for Eqs. (A22a) and (23a)

$\frac{1}{2}\, d^2C/dt^2$ for Eqs. (A22a) and (23b)

0 for Eqs. (A22b) and (23a)

$\frac{1}{4}\, d^2C/dt^2$ for Eqs. (A22b) and (23b)

$\frac{1}{96}\, d^2C/dt^2 - h^{-2}C$ for Eqs. (A22c) and (23a)

$\frac{13}{96}\, d^2C/dt^2 - h^{-2}C$ for Eqs. (A22c) and (23b)

$\frac{1}{16}\, d^2C/dt^2 - 2h^{-2}C$ for Eqs. (A22d) and (23a)

$\frac{1}{4}\, d^2C/dt^2$ for Eqs. (A22d) and (23b)

The results are used to check Eq. (24).

APPENDIX IV

DIAGONAL FORM OF A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

The system of differential equations whose stability is studied is given by

$$L\vec{y}^{\,\cdot} + M\vec{y} + \vec{f}(t) = 0 \tag{A23}$$

Here $\vec{y}$ and $\vec{f}$ denote respectively the dependent variable and a known vector with n components. L and M are n by n matrices. It is assumed that the determinant of L does not vanish. In writing our equations we consider the vectors as n by 1 or 1 by n matrices. The dot denotes differentiation with respect to the independent variable t. The matrices L and M are independent of t. (This assumption is always made in stability discussions. Heuristically, it is justified by the fact that for an arbitrarily fine mesh in an equation with variable coefficients one can carry out a considerable number of integration steps before the matrices L and M change appreciably.) We associate with Eq. (23) the generalized eigenvalue problem

$$(M - \lambda_k L)T_k = 0 \tag{A24}$$

and its adjoint

$$S_\ell (M - \lambda_\ell L) = 0 \tag{A25}$$

where T_k is a n by 1 matrix (i.e. a column vector) and S_ℓ is a 1 by n matrix (a row vector), λ_k and λ_ℓ are respectively the k^{th} and ℓ^{th} eigenvalue. Proceeding in a familiar manner one has

$$S_\ell M T_k - \lambda_k S_\ell L T_k = 0$$

$$S_\ell M T_k - \lambda_\ell S_\ell L T_k = 0$$

Hence

$$S_\ell L T_k = 0 \qquad \lambda_\ell \neq \lambda_k$$
$$S_\ell M T_k = 0$$

Let T and S be n by n matrices; the k^{th} column of T is given by T_k, the ℓ^{th} row of S by S_ℓ. Assume that S and T are normalized so that

$$SLT = I_n \tag{A26}$$

where I_n is the n dimensional identity matrix. Then one obtains from Eq. (A24)

$$SMT = \Lambda \tag{A27}$$

where Λ is a diagonal matrix whose k^{th} element is λ_k. Now set

$$\begin{aligned} y &= Tu \\ \dot{y} &= T\dot{u} \end{aligned} \tag{A28}$$

where u is a column vector with n components. Then one obtains from Eq. (A23) and by premultiplication with S and by using Eqs. (A26) and (A27)

$$\dot{u} + \Lambda u + Sf(t) = 0 \tag{A29}$$

This is a system of single equations, for Λ is a diagonal matrix.

This fact allows one to discuss the stability of the system in terms of the stability of a single scalar equation. Specifically, we shall derive the fact, familiar from the theory of integration process, that one obtains the same result if one first diagonalizes the system of equations and then applies a predictor corrector method or first applies the predictor corrector method and then diagonalizes. The same holds for a finite element procedure applied to the time.

In a predictor corrector method, no matter which specific scheme one applies, one always has as predictor formula

$$y_{i+1}^{(0)} = y_i + \beta_0 \dot{y}_i + \alpha_{-1} y_{i-1} + \beta_{-1} \dot{y}_{i-1} + \alpha_{-2} y_{i-2} + \beta_{-2} \dot{y}_{i-2} + \cdots$$

The values of the vector y after the ℓ^{th} corrector iteration is characterized by a superscript. The values of y and $y^{\cdot}$ at the point i and preceding points are fixed, therefore no reference to an iteration step is needed. One has in the corrector step

$$y_{i+1}^{(\ell+1)} = y_i + \delta_1 \dot{y}_{i+1}^{(\ell)} + \delta_0 \dot{y}_i + d_{-1} y_{i-1} + \delta_{-1} \dot{y}_{i-1} + d_{-2} y_{i-2} + \delta_{-2} \dot{y}_{i-2} + \cdots$$

The α_j's, β_j's, δ_j's and d_j's are constants independent of i. In addition, one has Eq. (A23) to be satisfied at the individual grid points. Substituting Eq. (A28) into the last two equations and premultiplying by T^{-1}, one obtains the same expressions but with y and $y^{\cdot}$ replaced by u and $u^{\cdot}$. In addition, one has Eq. (A29). In the resulting equations the vectors u_j and $u^{\cdot}_j$ are multiplied by constants or by the diagonal matrix Λ. The system can therefore be decomposed into its individual components. The further discussion can therefore be carried for each component in Eq. (A29) separately each with only one independent variable. The argument for implicit methods is the same.

To apply a finite element method to the solution of Eq. (A23) one multiplies this equation with scalar weight functions $w_m(t)$ (usually of finite support) and integrates with respect to t.

$$\int_{t_1}^{t_2} w_m(t) L\dot{y}(t)\,dt + \int_{t_1}^{t_2} w_m(t) M y(t)\,dt + \int_{t_1}^{t_2} w_m(t) f(t)\,dt = 0$$

Now we substitute Eqs. (A28), premultiply the resulting equation by S and apply Eqs. (A26) and (A27). One obtains

$$\int_{t_1}^{t_2} w_m(t) \dot{u}(t)\,dt + \int_{t_1}^{t_2} w_m(t) \Lambda u(t)\,dt + \int_{t_1}^{t_2} w_m(t) S f(t)\,dt = 0 \qquad \text{(A30)}$$

This expression is identical with the one which one would have obtained by applying the weight function $w_m(t)$ directly to Eq. (A29). Since Λ is a diagonal matrix, one can discuss each component of u separately. This result is the basis of the analysis in Section VI.

APPENDIX V

FUNDAMENTAL SOLUTIONS AND RELATED SUBJECTS

In this appendix the wave equation is regarded as the linearized equation for a supersonic flow at Mach number $\sqrt{2}$. We consider accordingly

$$-\phi_{xx} + \phi_{yy} = 0 \tag{A31}$$

The perturbation velocities are then given by the vector grad ϕ. We determine, also, the perturbation of the mass flow vector. In the nonlinearized problem the mass flow vector is given by ρ grad ϕ, where ρ depends upon grad ϕ.

Specifically, one has from Bernoullis, equation

$$dp + \rho\, d(\tfrac{1}{2}(\Phi_x^2 + \Phi_y^2)) = 0$$

and therefore, with the sound velocity "a" given by

$$a^2 = dp/d\rho$$

$$d\rho = -a^{-2}\rho(\Phi_x\, d\Phi_x + \Phi_y\, d\Phi_y)$$

Now the total potential Φ is given by

$$\Phi = Ux + \phi$$

Hence, for the x component in the perturbation of the mass flow

$$\rho_0 \phi_x (1 - U^2/a^2)$$

and for the y component

$$\rho_0 \phi_y$$

Here ρ_0 is the density in the unperturbed flow. One therefore obtains for the x and y components of the perturbation of the mass flow vector at Mach number $\sqrt{2}$, respectively $-\rho_0\phi_x$ and $\rho_0\phi_y$. (Important is the negative sign in the x component.) The potential equation in the form

$$\rho_0(-\phi_{xx} + \phi_{yy}) = 0$$

expresses the conservation of mass. By the usual integration by parts (Gauss' theorem) one obtains

$$\iint_R \rho_0(-\phi_{xx} + \phi_{yy})\,dx\,dy = -\rho_0 \oint_{\partial R} (\phi_x\, dy + \phi_y\, dx) \tag{A32}$$

where one travels along the path of integration in the direction for which the region R is to the left.

Assume that some portion of the contour is given by a left going characteristic $y = c + x$. Then the correction to the mass flow through this part of the contour is given by

$$-\rho_0 \int_{s_1}^{s_2} (\phi_x + \phi_y) \frac{\sqrt{2}}{2} ds$$

where ds is the line element. In this case

$$\frac{d\phi}{ds} = \phi_x \frac{dx}{ds} + \phi_y \frac{dy}{ds} = \frac{\sqrt{2}}{2} (\phi_x + \phi_y)$$

Thus, one obtains as a correction to the mass flow passing through a left going characteristic

$$-\rho_0 \int_{s_1}^{s_2} \frac{d\phi}{ds} ds = -\rho_0 (\phi_{s_2} - \phi_{s_1}) \quad \text{(A33)}$$

Analogously, for a right going characteristic

$$-\rho_0 \int_{s_1}^{s_2} (\phi_x - \phi_y) \frac{\sqrt{2}}{2} ds = \rho_0 (\phi(s_2) - \phi(s_1)) \quad \text{(A34)}$$

Eq. (A31) is solved by

$$\phi = f_1(y+x) + f_2(y-x)$$

To obtain a fundamental solution we postulate that $\phi \equiv 0$ for $x \leq 0$ and that at $x = 0$, ϕ_x is given by the negative of a delta function at the origin. The requirement $\phi \equiv 0$ at $x = 0$ is satisfied by choosing

$$f_1 = -f_2 = f$$

Then one has

$$\phi_x = f'(y+x) + f'(y-x)$$

It follows that

$$f(z) = 0 \quad \text{for } z < 0$$
$$f(z) = -\tfrac{1}{2} \quad \text{for } z > 0$$

Thus, one finds for $x > 0$ as the potential caused by a source at the origin

$$\left.\begin{array}{ll} \phi = 0 & y < -x \\ \phi = -1/2 & -x < y < x \\ \phi = 0 & y > x \end{array}\right\} \quad x > 0 \quad \text{(A35)}$$
$$\phi = 0 \qquad x < 0$$

If one crosses the right going characteristic $y = -x$ along a left going characteristic, then the correction to the mass flow is according to Eq. (A33) $(1/2)\rho_0$. The same result is obtained if one corrects the characteristic $y = x$. A fundamental solution gives a correction to the mass flow vector along the characteristics through the origin in the direction of the characteristics in the form of a delta function of intensity $1/2\rho_0$. Everywhere else the correction is zero (Figure 30.). The gradient of the velocities is perpendicular to these characteristics. The perturbation does not die out with distance.

Except for a constant, the fundamental solution for the Laplace equation is given by $\log(x^2 + y^2)$. The corresponding expression for the wave equation is given by $\log(x^2 - y^2)$. For $x \neq 0$, $y \neq 0$ and $x \neq |y|$ this expression certainly satisfies the partial differential equation, but obviously it does not give the desired fundamental solution.

Also of interest is the perturbation caused in an axisymmetric flow by a source at the origin. It provides the fundamental solution for the three dimensional wave equation. The linearized equation for three dimensional supersonic flows for Mach number $\sqrt{2}$ is given by

$$-\phi_{xx} + \phi_{yy} + \phi_{zz} = f(x,y,z) \qquad \text{(A36)}$$

where f(x,y,z) is the local source strength. One obtains, by integration over a volume R

$$\iiint_R (-\phi_{xx} + \phi_{yy} + \phi_{zz})\,dx\,dy\,dz = \iiint_R f(x,y,z)\,dx\,dy\,dz$$

The right hand sides gives the combined strength of the sources within the volume R. Now we assume axial symmetry and introduce cylindrical coordinates

$$x \rightarrow x$$
$$r^2 = y^2 + z^2$$

One then obtains

$$I = 2\pi \iint_R \left[\frac{\partial}{\partial r}(r\phi_r) - \frac{\partial}{\partial x}(r\phi_x)\right] dx\,dr = 2\pi \iint_R f(x,r)\,dx\,dr \qquad \text{(A37)}$$

It is convenient to introduce coordinates oriented according to the directions of the characteristics (Fig. 31)

$$\xi = r + x$$
$$\eta = -r + x$$
$$r = \frac{\xi - \eta}{2}$$
$$x = \frac{\xi + \eta}{2}$$

with the restriction

$$r \geq 0 \; ; \quad \xi \geq \eta$$

The x axis is given by $\xi = \eta$.

One obtains for the Jacobian of the transformation

$$\frac{d(\xi,\eta)}{d(r,x)} = 2$$

Moreover

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \; ; \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$$

Thus, one obtains the following expression for the total mass flow.

$$J = -\pi \iint_R \left\{ \frac{\partial}{\partial \xi}\left[(\xi-\eta)\frac{\partial \phi}{\partial \eta}\right] + \frac{\partial}{\partial \eta}\left[(\xi-\eta)\frac{\partial \phi}{\partial \xi}\right]\right\} d\xi \, d\eta \qquad \text{(A38)}$$

In these coordinates the original equation for the perturbation potential, namely

$$-\frac{\partial}{\partial x}(r\phi_x) + \frac{\partial}{\partial r}(r\phi_r) = 0$$

assumes the form

$$\frac{\partial}{\partial \xi}\left[(\xi-\eta)\frac{\partial \phi}{\partial \eta}\right] + \frac{\partial}{\partial \eta}\left[(\xi-\eta)\frac{\partial \phi}{\partial \xi}\right] = 0 \qquad \text{(A39)}$$

Here one has the particular solution

$$\phi = -\frac{1}{2\pi}(x^2 - r^2)^{-1/2} = -\frac{1}{2\pi}\xi^{-1/2}\eta^{-1/2} \qquad \text{(A40)}$$

This is easily verified.

The expression Eq. (A38) can be transformed into a contour integral

$$J = \pi \oint (\xi-\eta)\phi_\eta \, d\eta - (\xi-\eta)\phi_\xi \, d\xi$$

The integrand vanishes at the x axis ($\xi = \eta$).

To determine the total source strength we choose for R a region bounded by the x axis ($\xi = \eta$), by a line $\xi = \xi_1$ and closed to the left by some arbitrary curve. Then one finds

$$I = -\pi \int_{\eta=-\varepsilon}^{\xi_1} (\xi_1 - \eta) \frac{\partial \phi}{\partial \eta} d\eta = -\pi \xi_1 \phi(\xi_1, \eta) \Big|_{\eta=-\varepsilon}^{\xi} + \pi \int_{\eta=-\varepsilon}^{\xi_1} \eta \frac{\partial \phi}{\partial \eta} d\eta$$

Substituting the specific expression for ϕ, Eq. (A40) one obtains

$$I = \pi \xi_1 \frac{1}{2\pi} \xi_1^{-1} + \pi \frac{1}{2\pi} \xi_1^{-1/2} \int_{\eta=-\varepsilon}^{\xi_1} \eta \cdot \frac{1}{2} \eta^{-1/2} d\eta = 1$$

Notice that the result is independent of ξ_1. Since the differential equation is satisfied inside of the cone $\xi > \eta$ there are no sources inside of this Mach cone and since the result is independent of ξ_1 there are no sources at the Mach cone.

The expression

$$-\pi \int_{-\varepsilon}^{\bar{\eta}} (\xi_1 - \eta) \frac{\partial \phi}{\partial \eta} d\eta = \frac{1}{2} \xi_1^{1/2} \bar{\eta}^{-1/2} + \frac{1}{2} \xi_1^{-1/2} \bar{\eta}^{-1/2}$$

gives the total outflow of mass due to the perturbation through the portion of the surface lying between $\eta = -\varepsilon$ and $\eta = \bar{\eta}$. The total outflow of mass up tends to infinity as $\bar{\eta} \to 0$. (For the two dimensional case one has a concentrated finite additional outflow along the corresponding characteristic.) For larger values of η the total additional outflow becomes smaller, at $\xi = \xi_1$ it reaches the value 1. The perturbation of the mass flow vector inside the cone therefore causes inside a reduction of the outflow. The expression (A40) is the fundamental solution for the three dimensional problem.

Its form is, of course, too singular to serve as the basis for a numerical procedure. The singularity is brought into a less severe form if one considers source distributions and carries out some of the necessary integrations beforehand.

Consider a three dimensional problem and assume that the source strength is constant along the z axis. In this manner one will obtain the two dimensional potential. Using Eq. (A40) one computes

$$\phi(x,y) = -\frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{Re}\, \frac{dz}{(x^2-y^2-z^2)^{1/2}} = -\frac{1}{2\pi}\int_{z=-(x^2-y^2)^{1/2}}^{(x^2-y^2)^{1/2}} \frac{dz}{(x^2-y^2-z^2)^{1/2}} = -\frac{1}{2}$$

This is indeed the result Eq. (A35).

To see what happens if the source density in the y direction is not constant, we consider an example with a periodic source density in the z direction. Consider first the elliptic problem

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0$$

Periodicity of the source density in the y direction implies periodicity of the potential. Accordingly, we set

$$\phi = \tilde{\phi}\, \exp(i\nu z)$$

One obtains

$$\tilde{\phi}_{xx} + \tilde{\phi}_{yy} - \nu^2\tilde{\phi} = 0$$

and in polar coordinates

$$\tilde{\phi}_{rr} + \frac{1}{r}\tilde{\phi}_r + \frac{1}{r^2}\tilde{\phi}_{\theta\theta} - \nu^2\tilde{\phi} = 0$$

Now we set

$$\tilde{\phi}(r,\theta) = \bar{\phi}(r)$$

Then one obtains

$$\bar{\phi}_{rr} + \frac{1}{r}\bar{\phi}_r - \nu^2\bar{\phi} = 0$$

This equation is solved by a linear combination of $H_0^{(1)}(i\nu r)$ and $H_0^{(2)}(i\nu r)$. For $r \to \infty$ the solution must tend to zero; obviously the effect of sources periodically distributed along the z axis will cancel at a large distance from the z axis. Hence

$$\bar{\phi} = \mathrm{const}\; H_0^{(1)}(i\nu r)$$

The constant is determined by the local source density at $r = 0$.

The supersonic problem is treated analogously. One has

$$-\tilde{\phi}_{xx} + \tilde{\phi}_{yy} - \nu^2\tilde{\phi} = 0$$

and with $r^2 = x^2 - y^2$ and $\tilde{\phi} = \bar{\phi}(r)$

$$\bar{\phi}_{rr} + \frac{1}{r}\bar{\phi}_r + \nu^2\bar{\phi} = 0$$

This equation is solved by

$$\Phi = J_0(\nu r) = J_0(\nu(x^2 - r^2)^{1/2})$$

Consider this expression at a fixed x. The argument of J_0 varies between 0 for the Mach cone and x for the x axis.

The variation is small if x is small. The jump of ϕ at the characteristic r = 0 is the same for all values of x; namely $J_0(0)$. For large values of x, the expression is an oscillatory function of y. This result has obvious implications for the choice of a grid in three-dimensional problems.

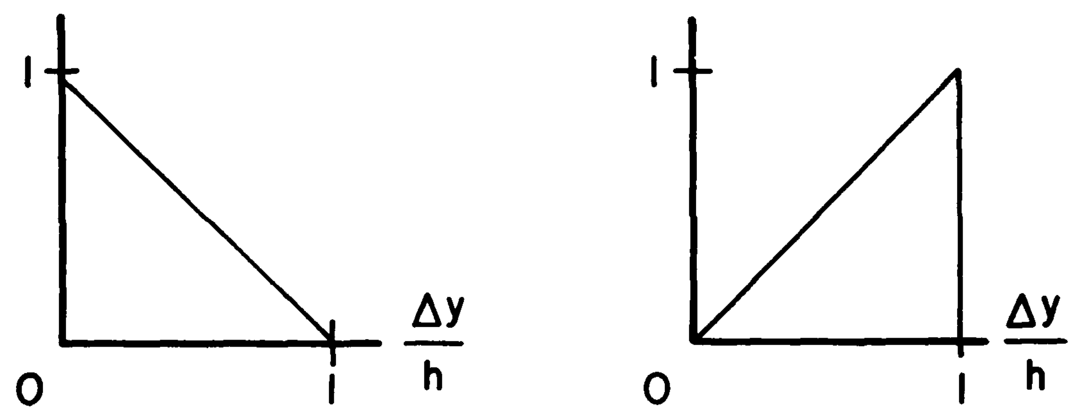


Figure 1. Linear Elemental Shape Functions.

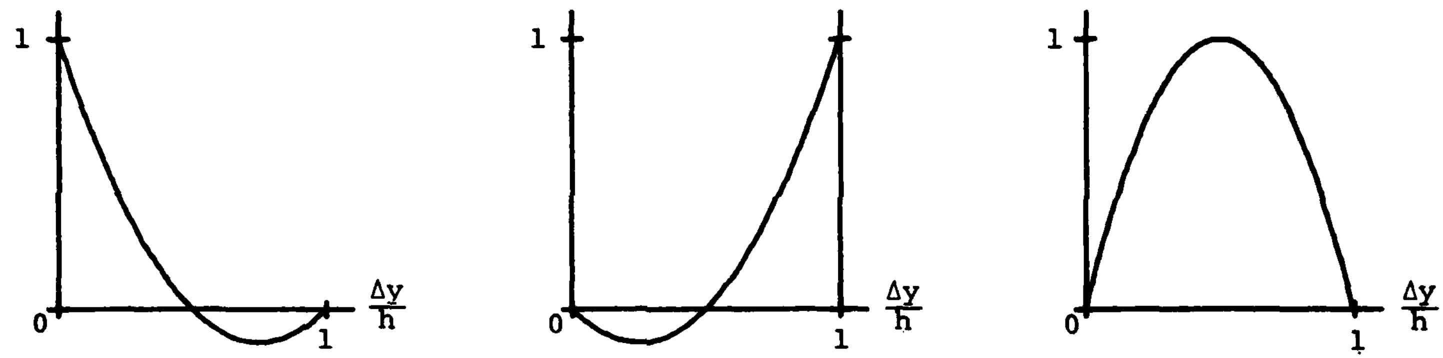


Figure 2. Quadratic Elemental Shape Functions.

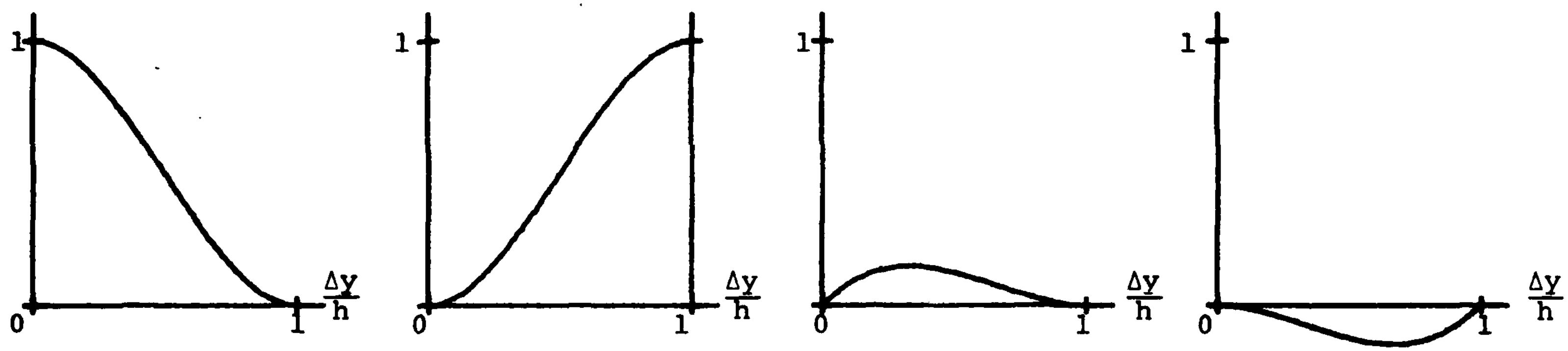


Figure 3. Third Degree Elemental Shape Functions.

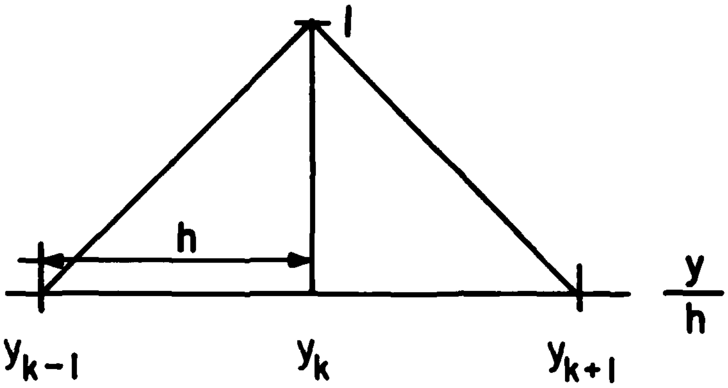


Figure 4. Shape Function and, in Case 2a, Weight Function Belonging to Point $y = y_k$.

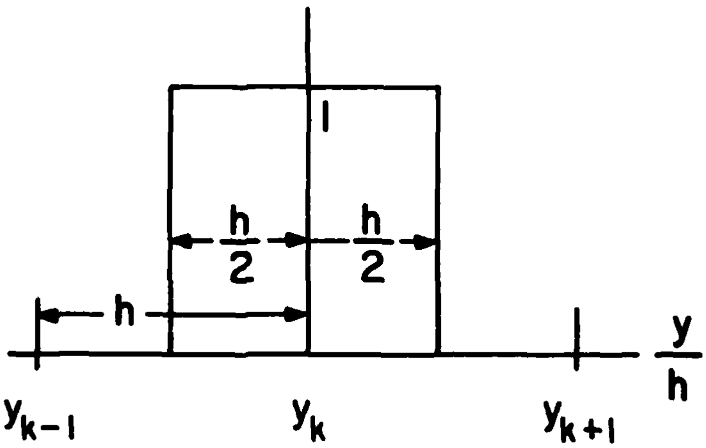
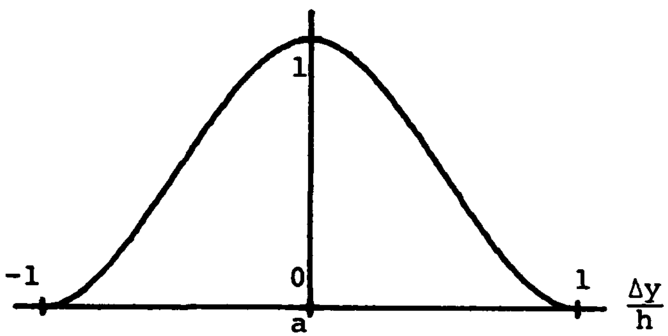


Figure 5. Weight Function Belonging to Case 2b.



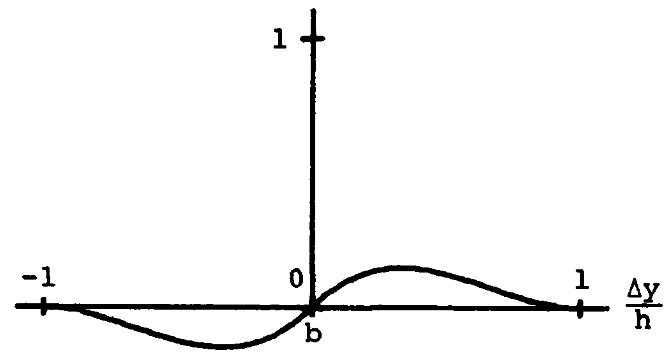


Figure 6. Third Degree Shape Functions and, for the Case 3a, Third Degree Weight Functions. Figure 6a is also the Weight Function for Case 5a.

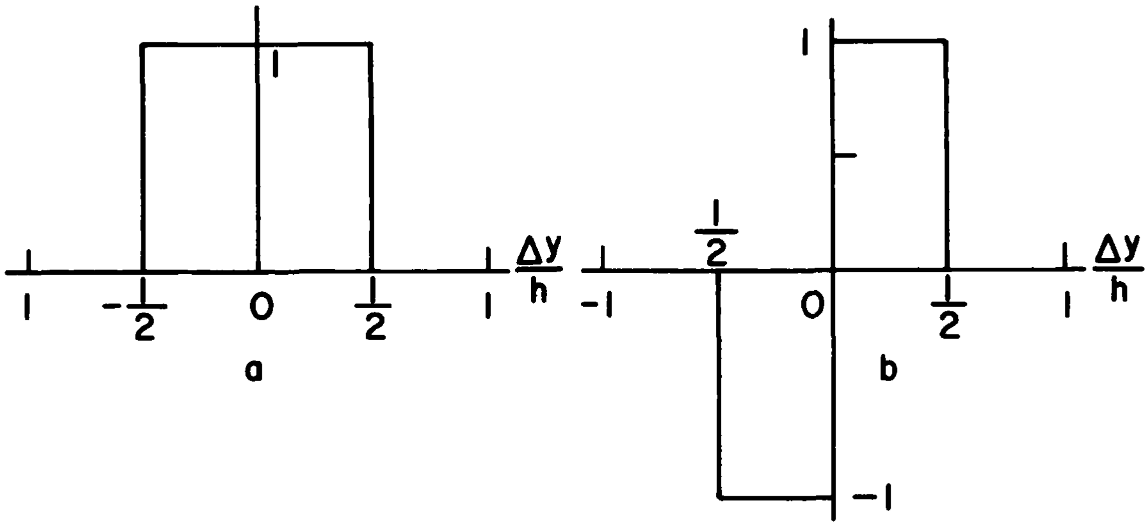


Figure 7. Weight Functions for Case 3b. Figure 7a is also the Weight Function for Cases 5b and 6b.

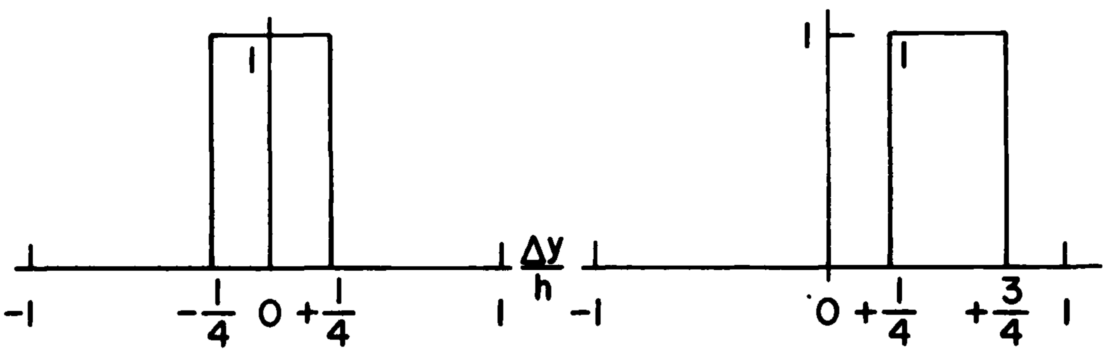


Figure 8. Weight Functions for the Cases 3c and 4b.

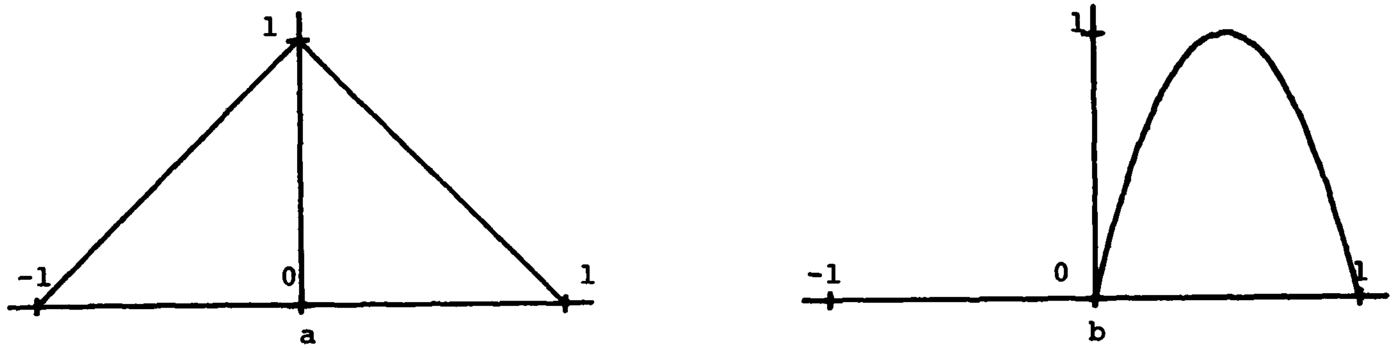


Figure 9. Second Degree Shape Functions, and for the Case 4a, Second Degree Weight Functions. Figure 9a is also the Weight Function for Case 6a.

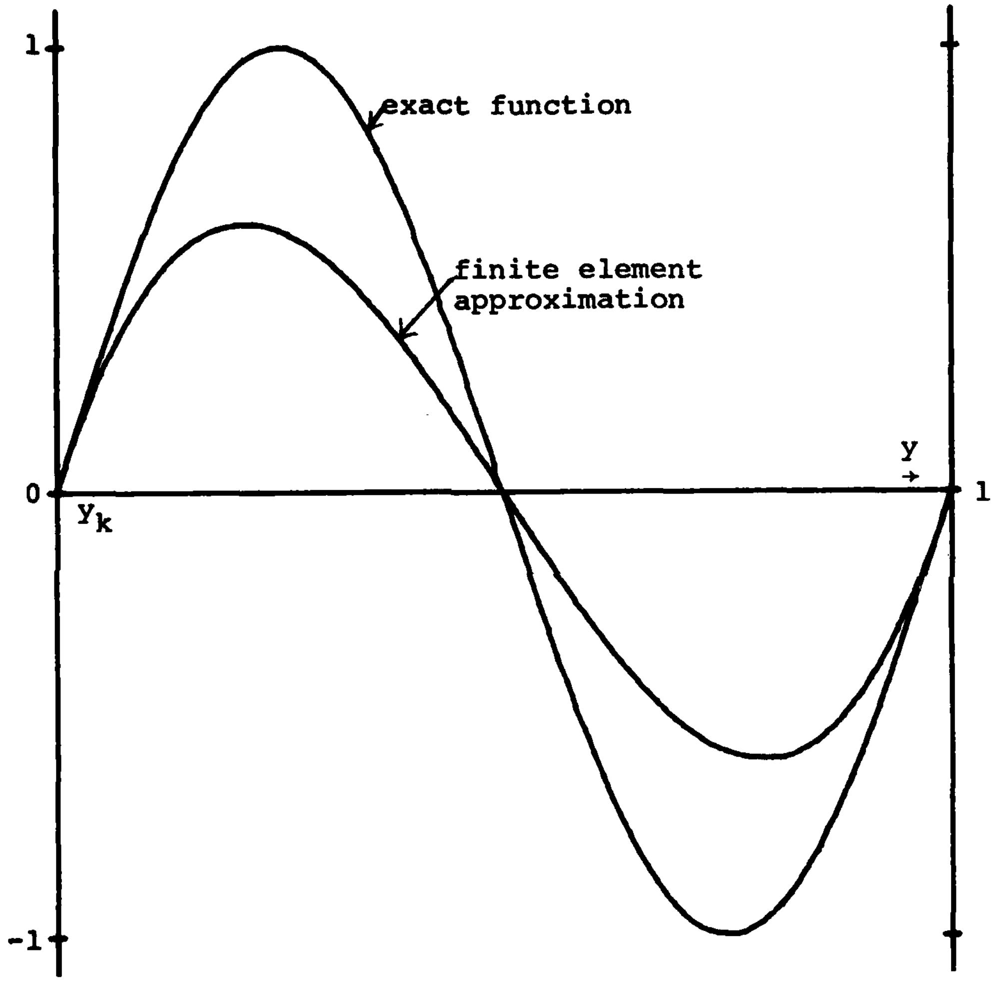


Figure 10. Approximation for ϕ obtained for $\mu = 0$ by $\phi_k = 0$, $\phi_k' = 1$ and exact function ϕ for $\mu = 2\pi$.

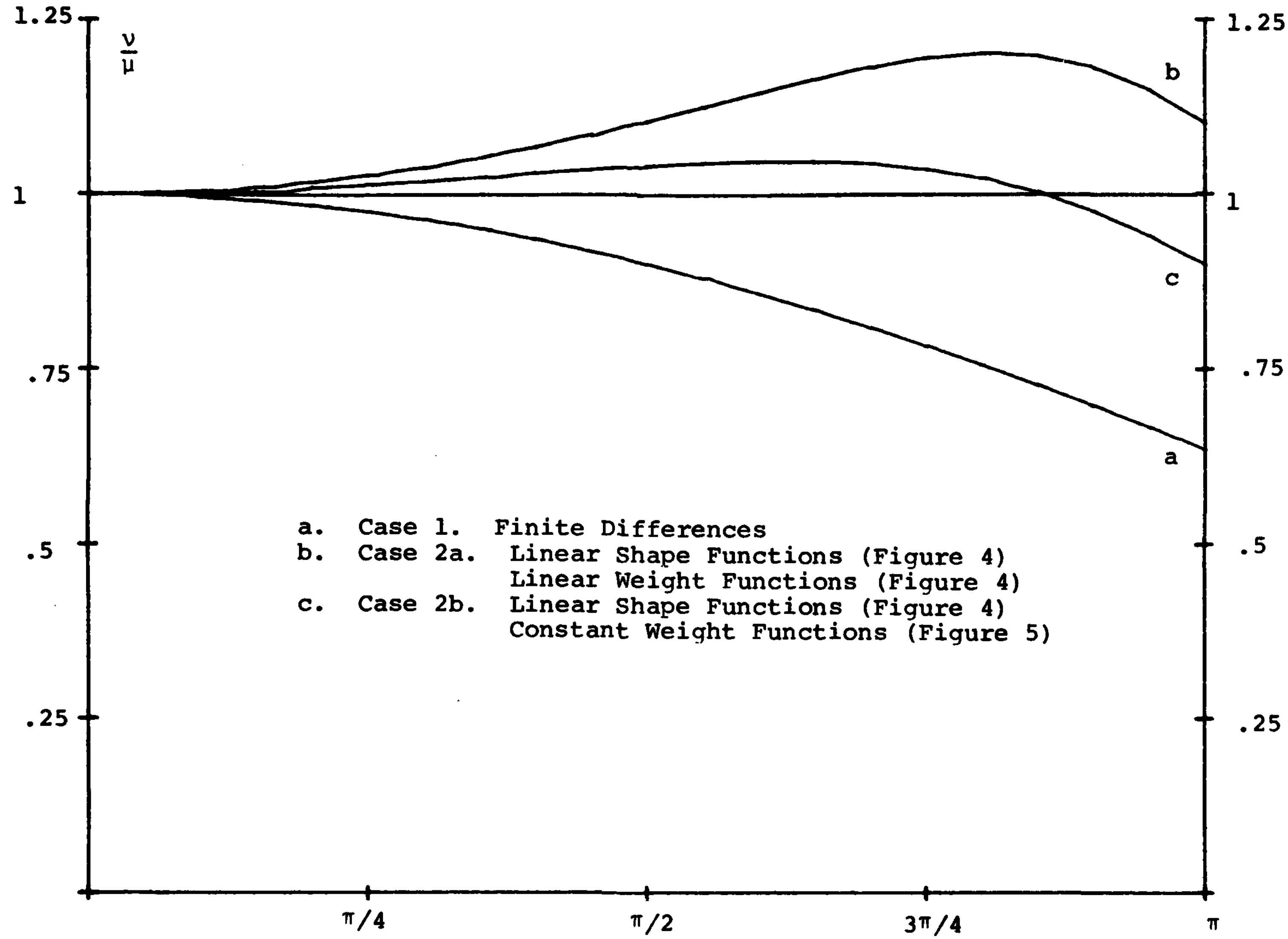


Figure 11. Ratio of Approximate and Exact Wave Velocities.

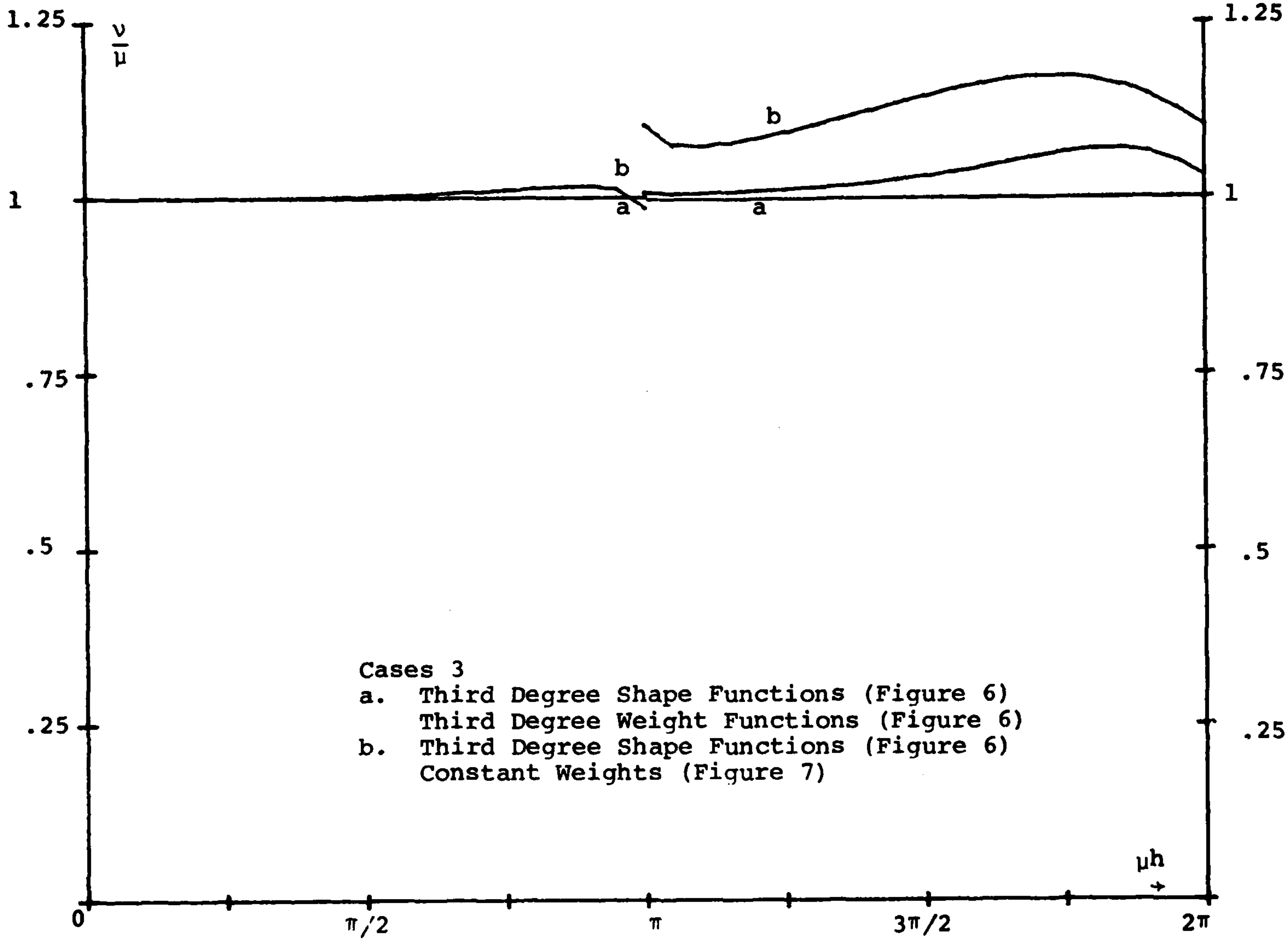


Figure 12. Ratio of Approximate and Exact Wave Velocities.

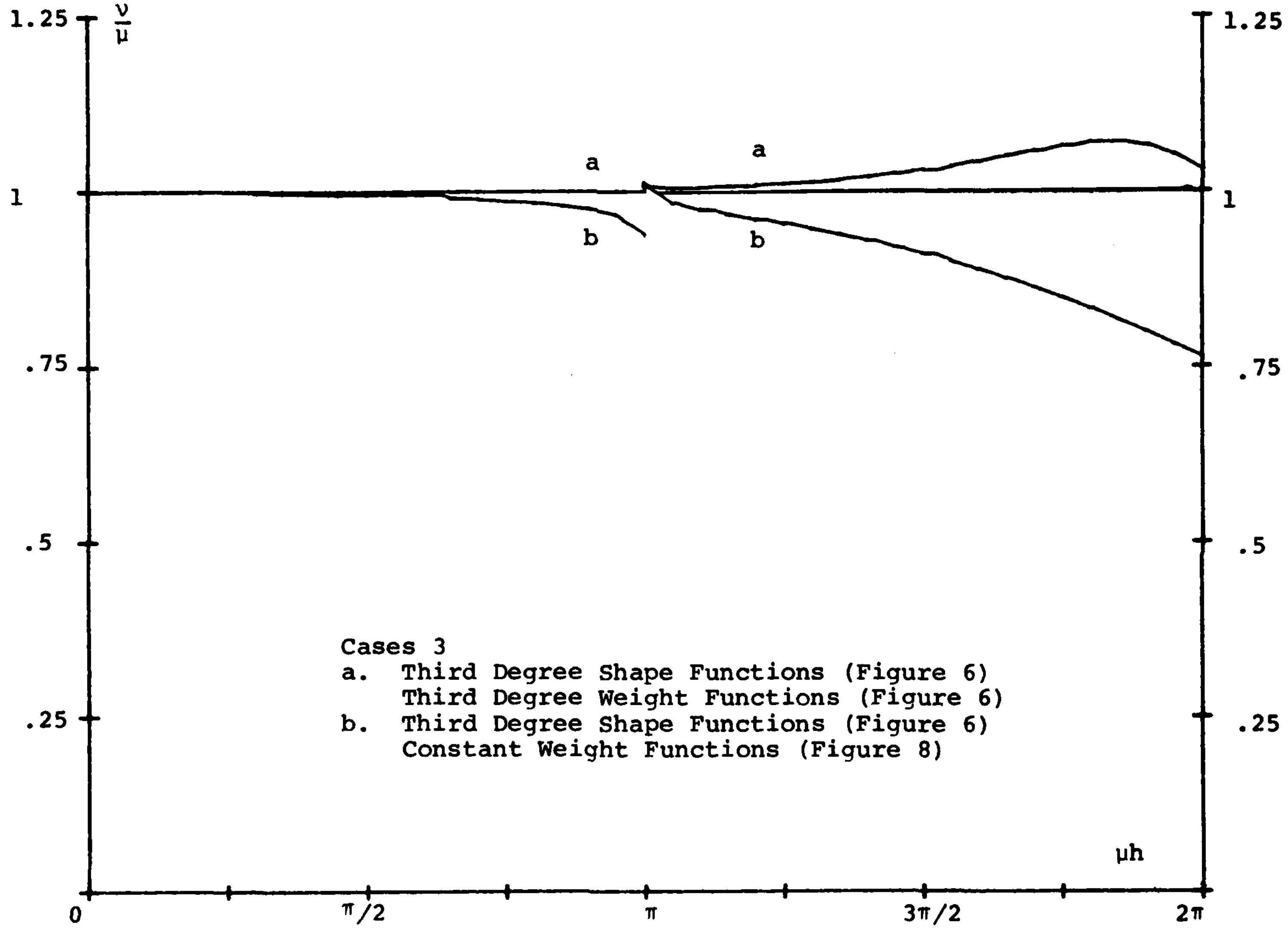


Figure 13. Ratio of Approximate and Exact Wave Velocities.

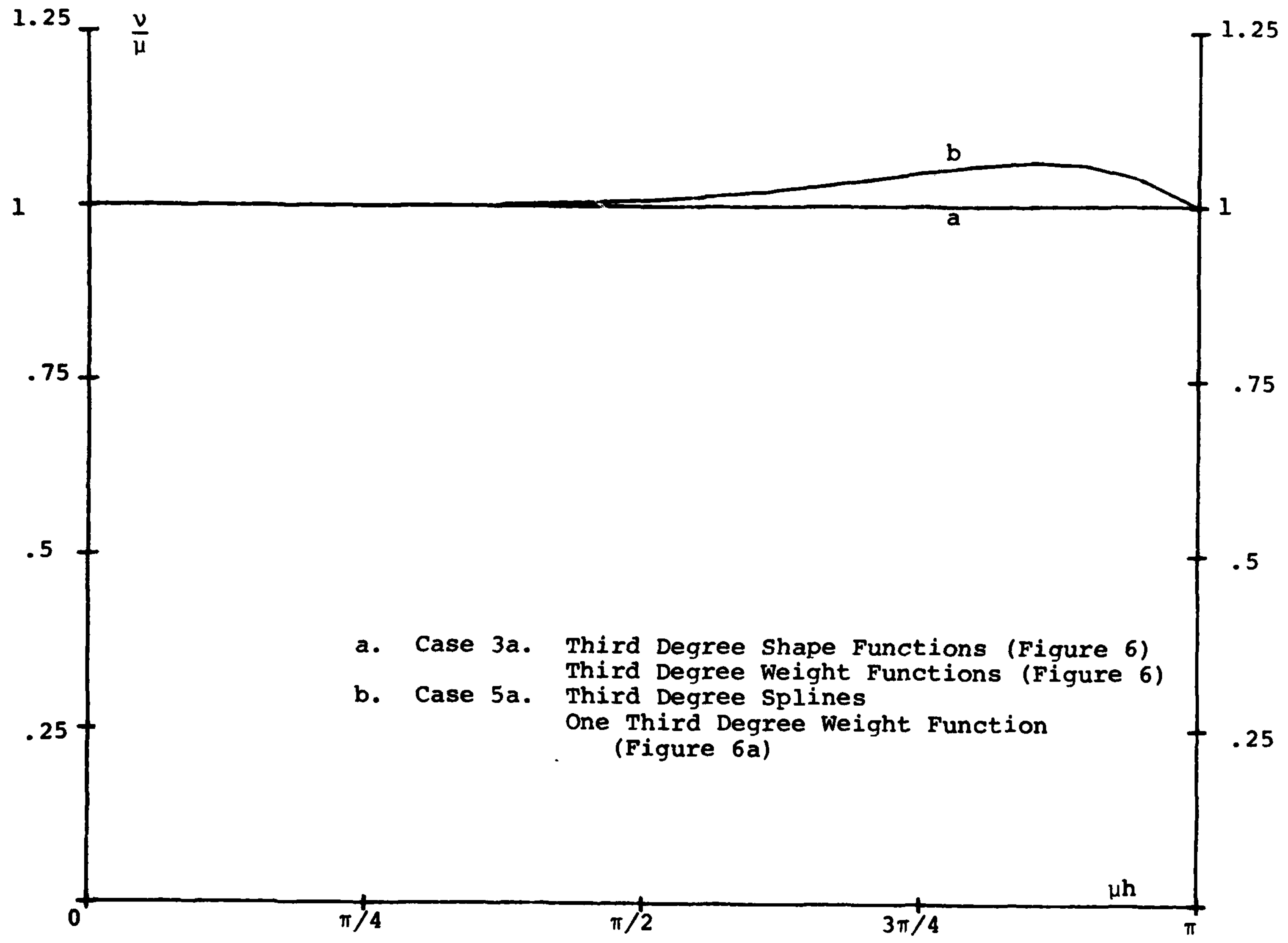


Figure 14. Ratio of Approximate and Exact Wave Velocities.

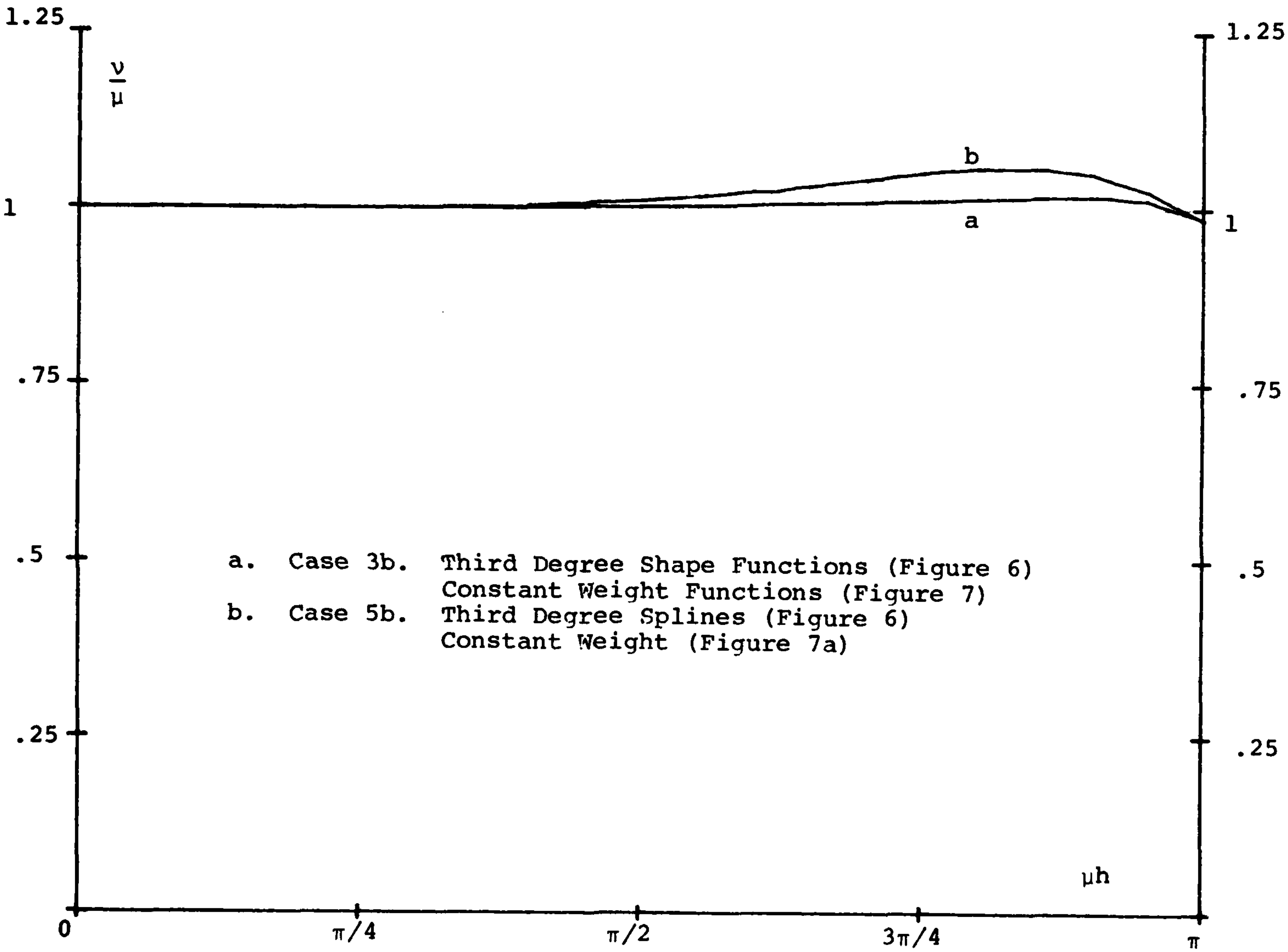


Figure 15. Ratio of Approximate and Exact Wave Velocities.

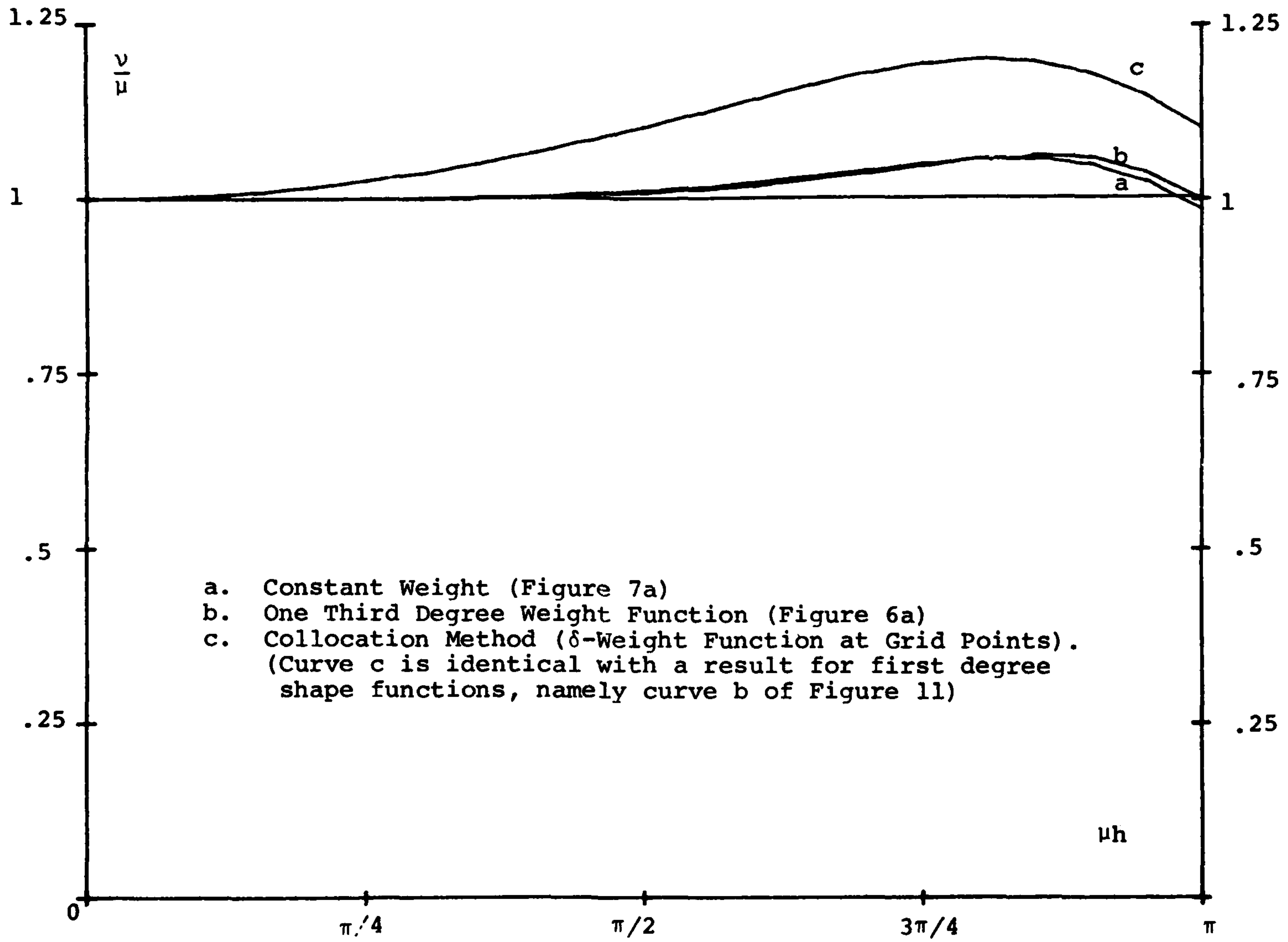


Figure 16. Ratio of Approximate and Exact Wave Velocities Comparison of Formulae Based on Third Degree Splines.

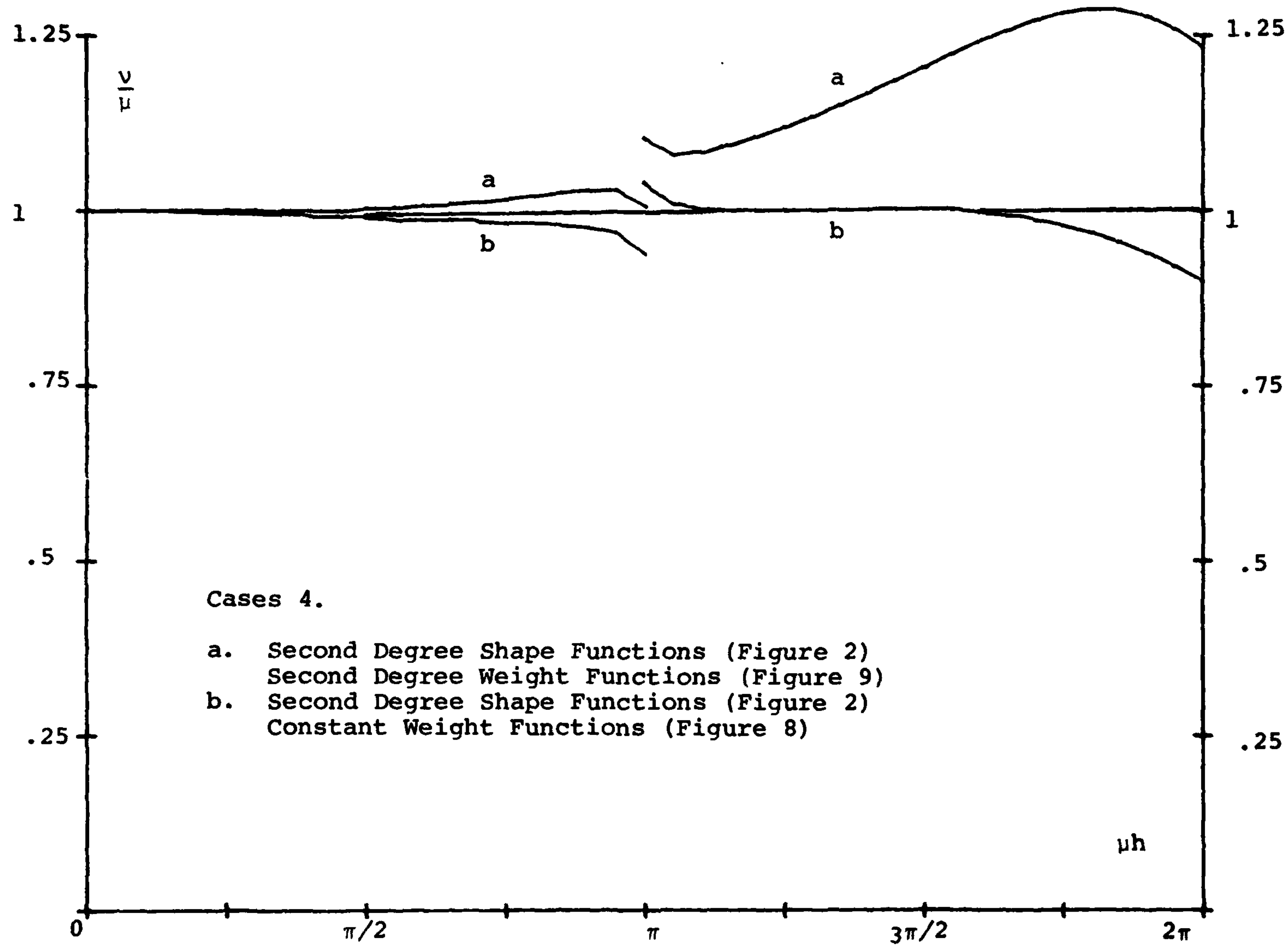


Figure 17. Ratio of Approximate and Exact Wave Velocities.

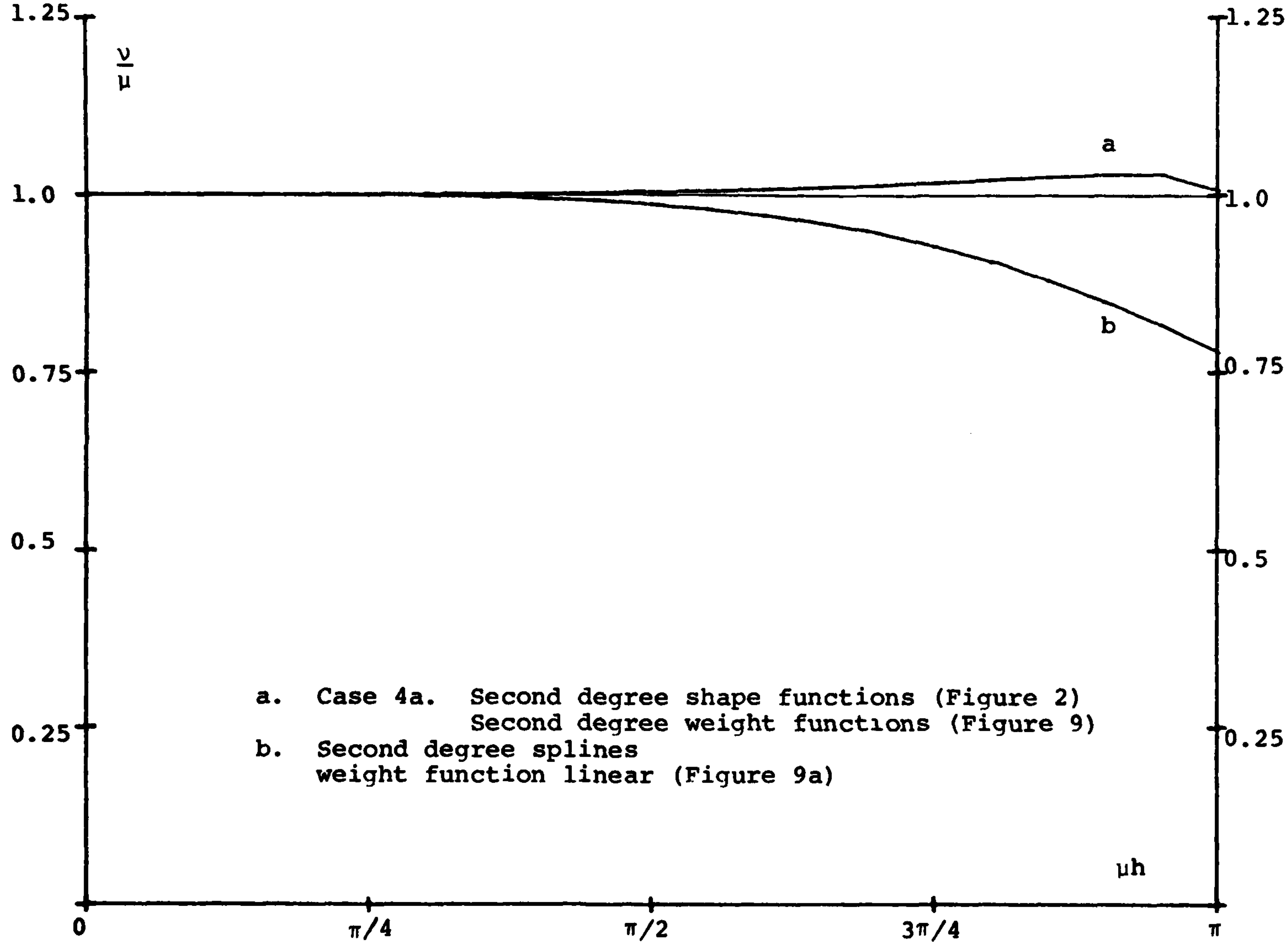


Figure 18. Ratio of Approximate and Exact Wave Velocities.

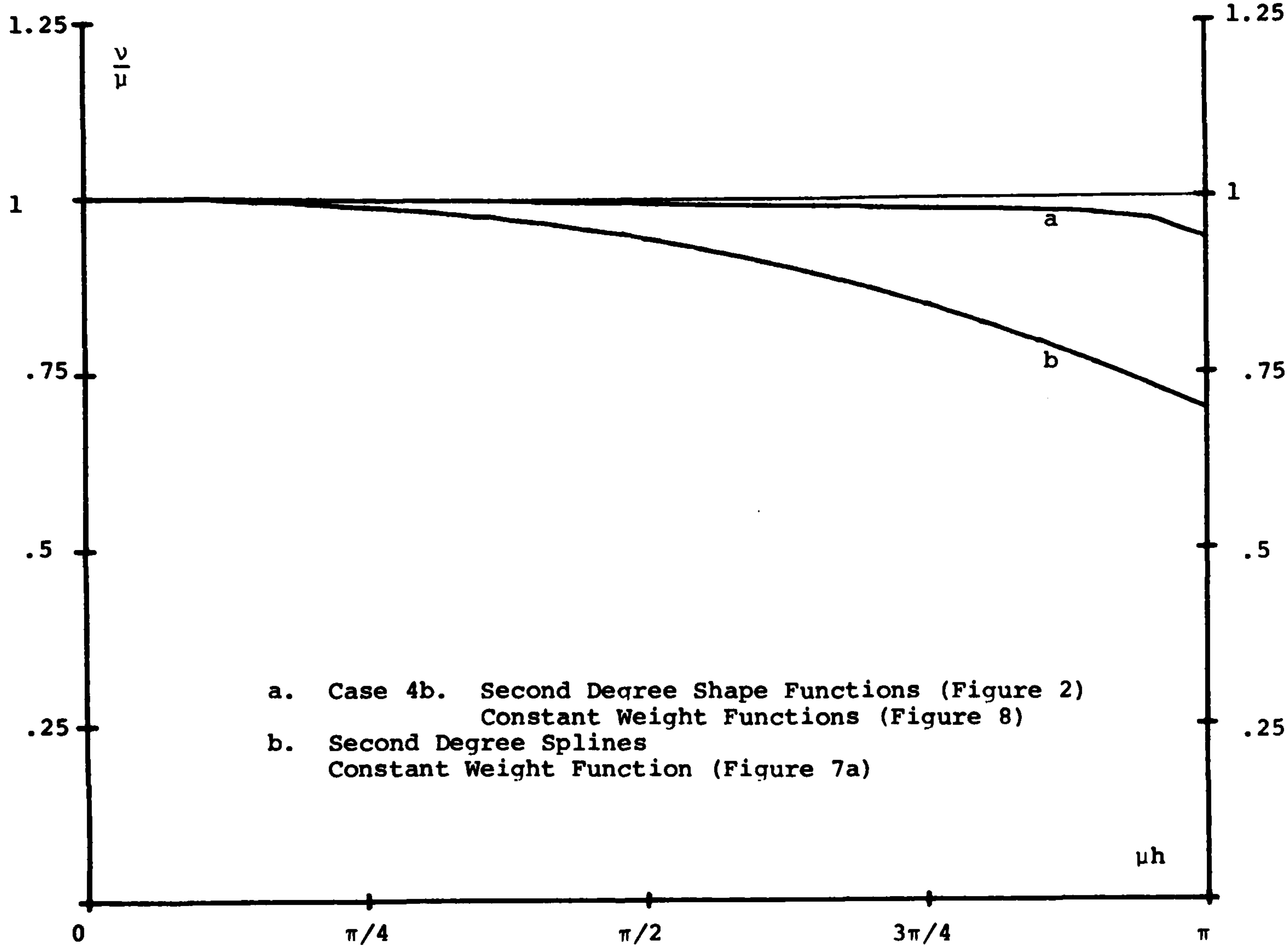


Figure 19. Ratio of Approximate and Exact Wave Velocities.

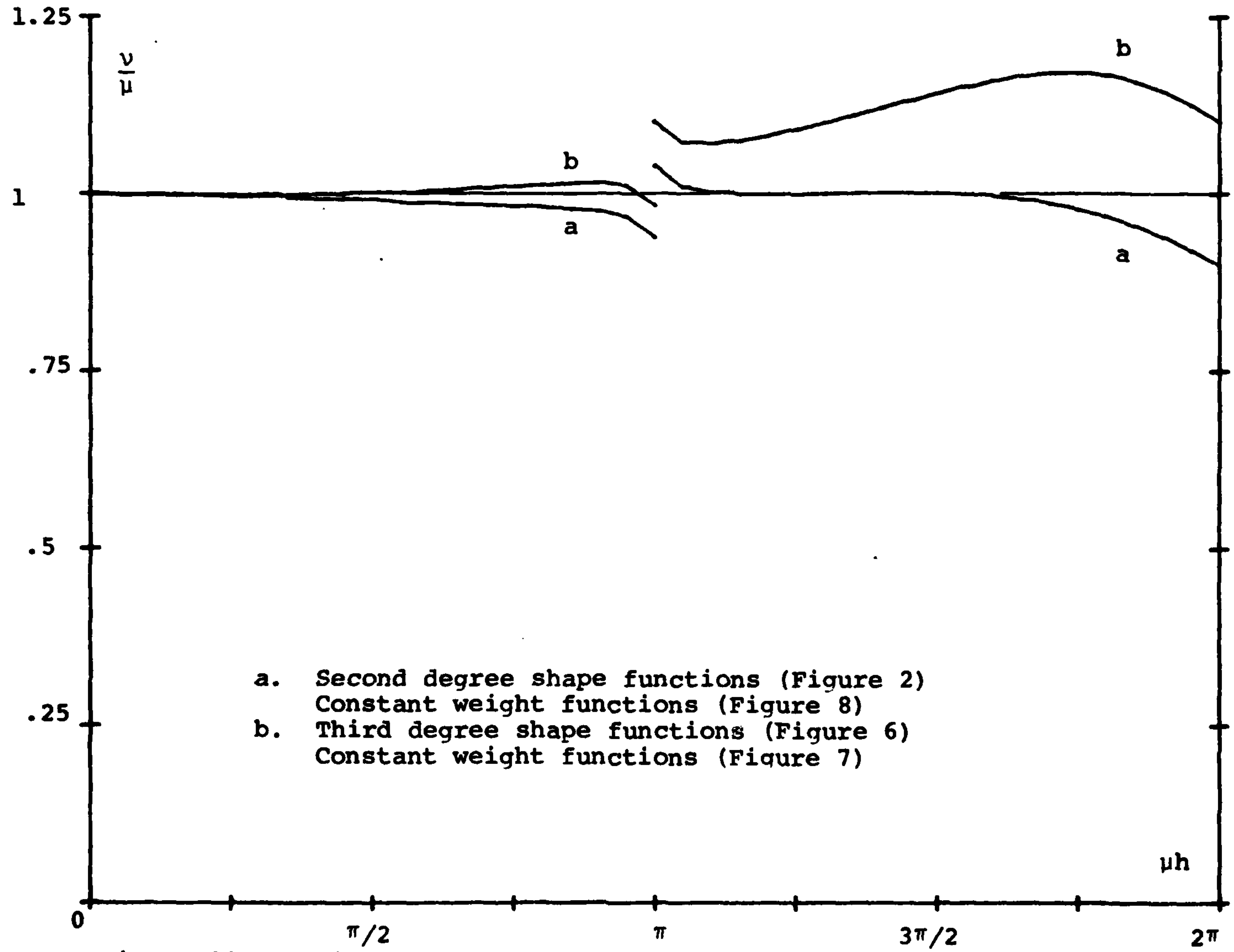


Figure 20. Ratio of Approximate and Exact Wave Velocities, Comparison Between Second Degree and Third Degree Shape Functions.

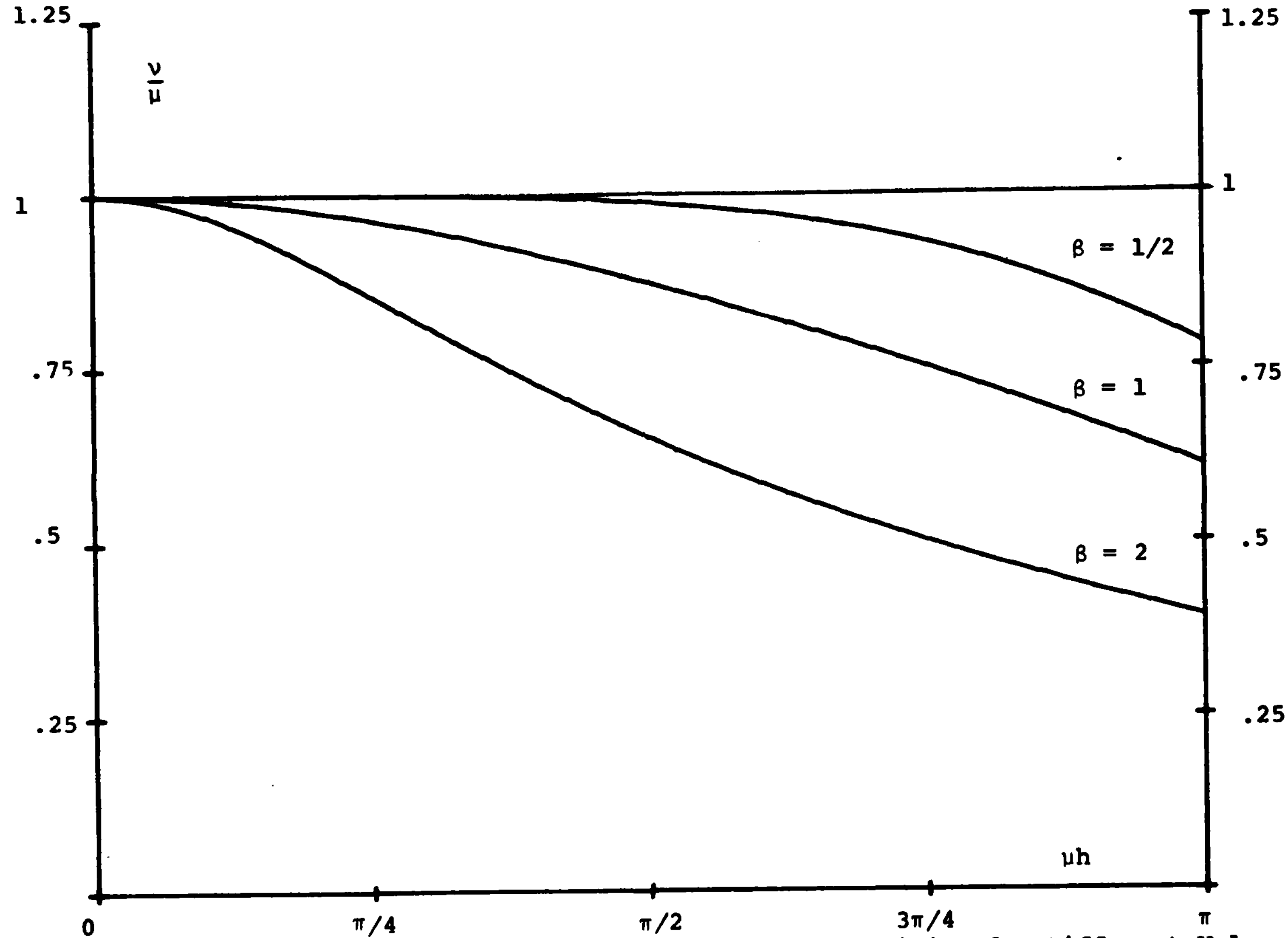


Figure 21. Ratio of Approximate and Exact Wave Velocities for Different Values of the Courant Number $\beta=h_t/h_s$ and Different Forms of the Denominators $1-\alpha \sin^2(\mu h/2)$ in Eq. (36), (37), and (38). $\alpha=0$.

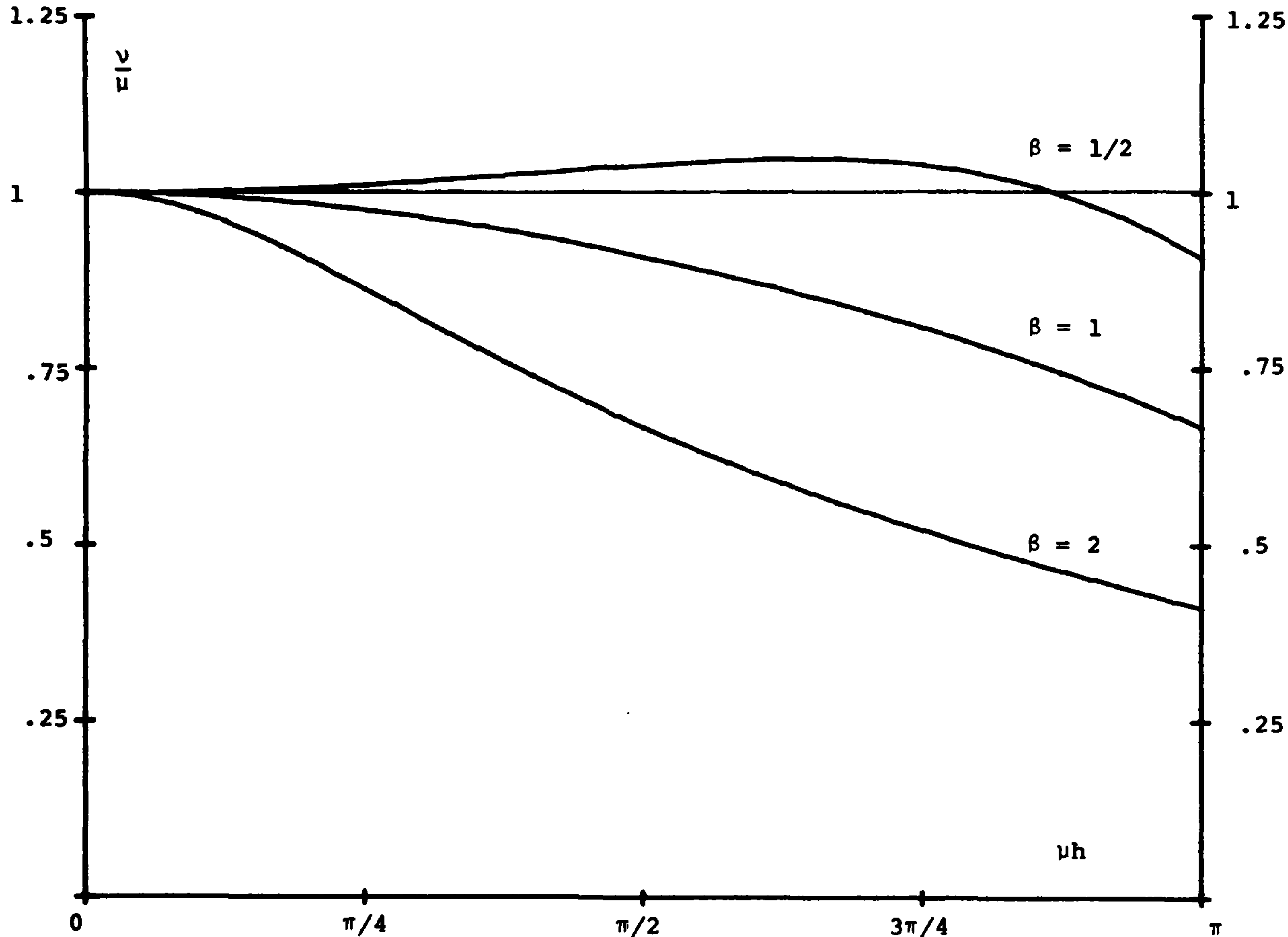


Figure 22. Ratio of Approximate and Exact Wave Velocities for Different Values of the Courant Number $\beta=h_t/h_s$ and Different Forms of the Denominator $1-\alpha \sin^2(\mu h/2)$ in Eqs. (36), (37) and (38). $\alpha=2/3$.

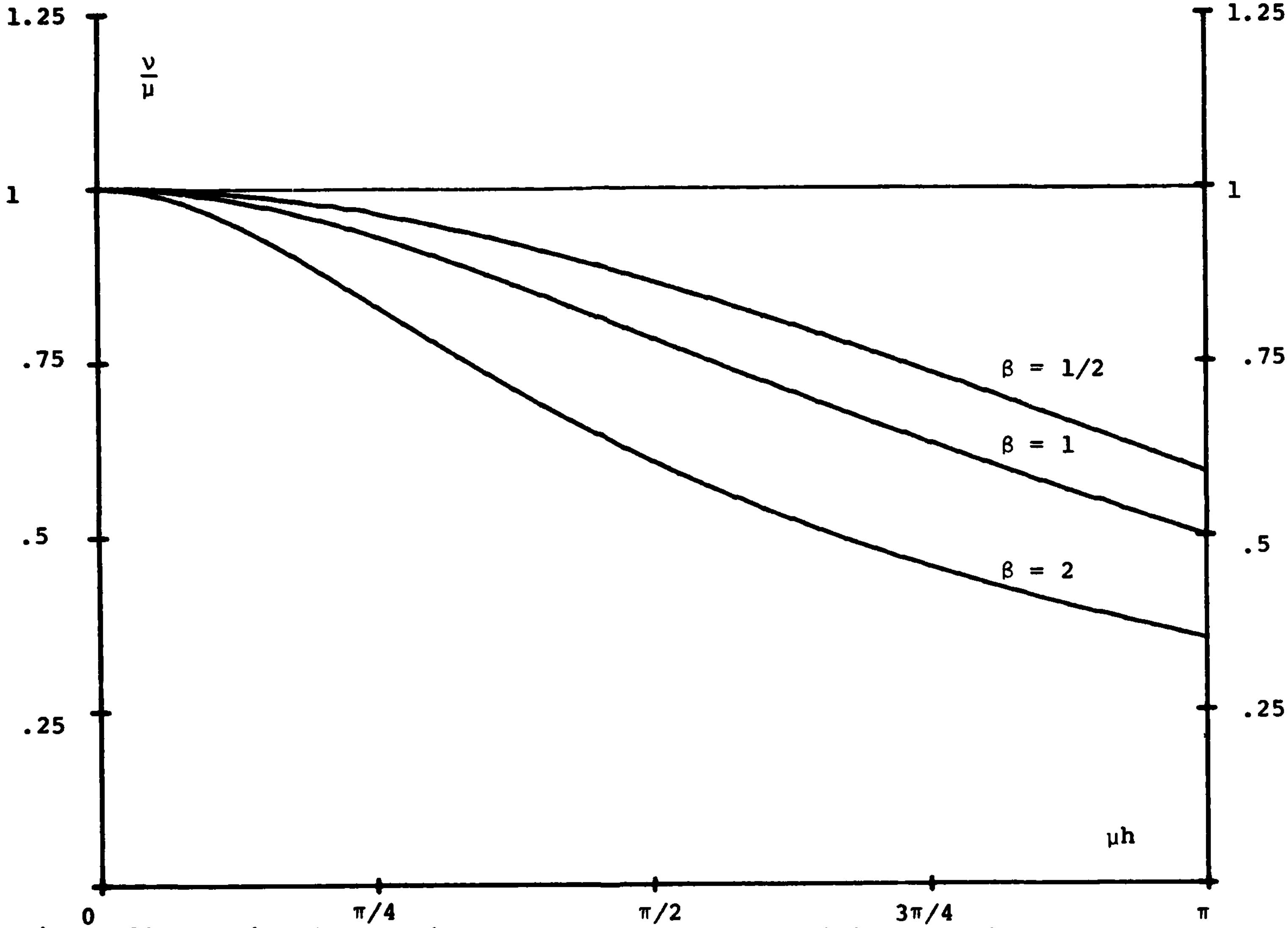


Figure 23. Ratio of Approximate and Exact Wave Velocities for Different Values of the Courant Number $\beta=h_t/h_s$ and Different Forms of the Denominators $1-\alpha \sin^2(\mu h/2)$ in Eqs. (36), (37) and (38). $\alpha=1/2$

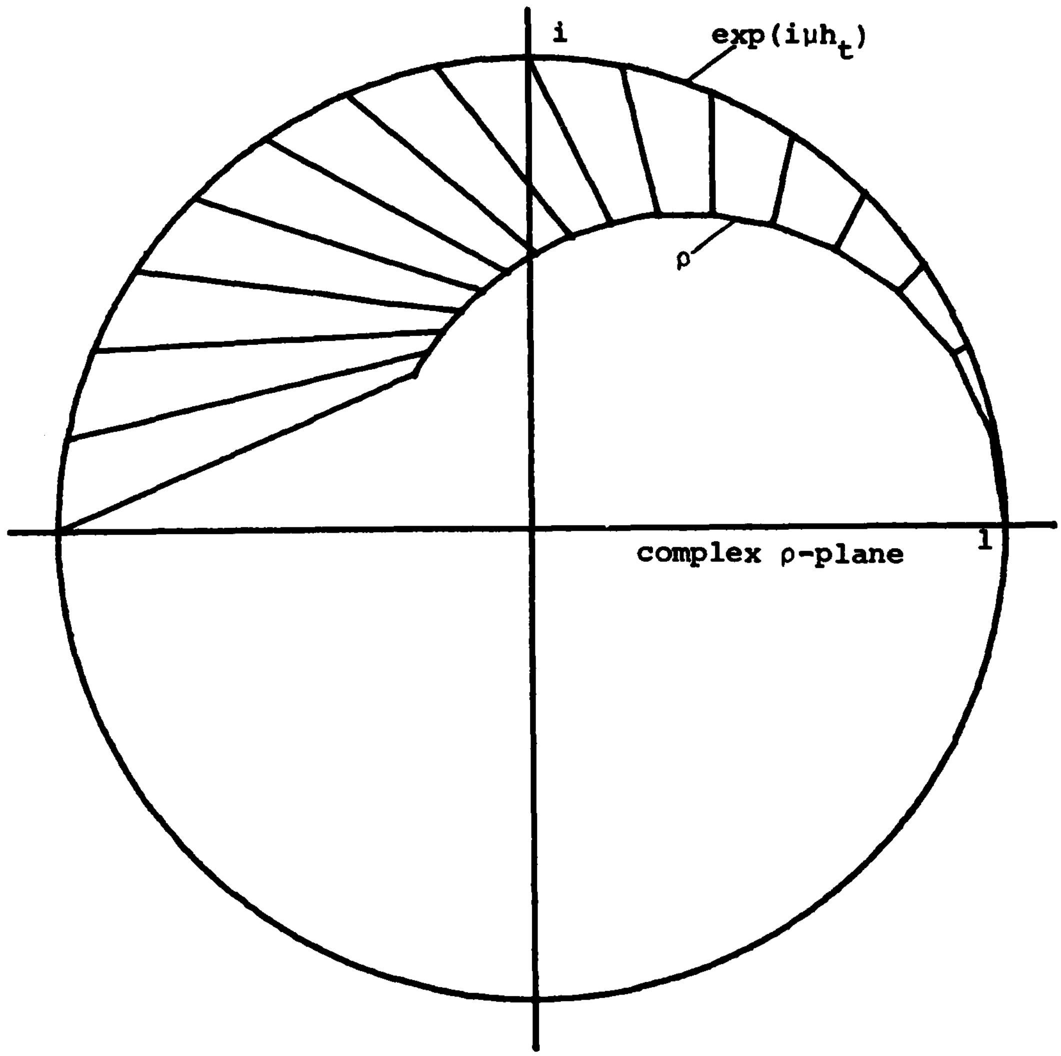


Figure 24. Amplification Factor ρ in the Complex ρ Plane and Corresponding Values of $\exp(i\mu h_t)$, for Linear Shape Functions and Constant Weight Between $t_k-\varepsilon$ and $t_{k+1}-\varepsilon$.

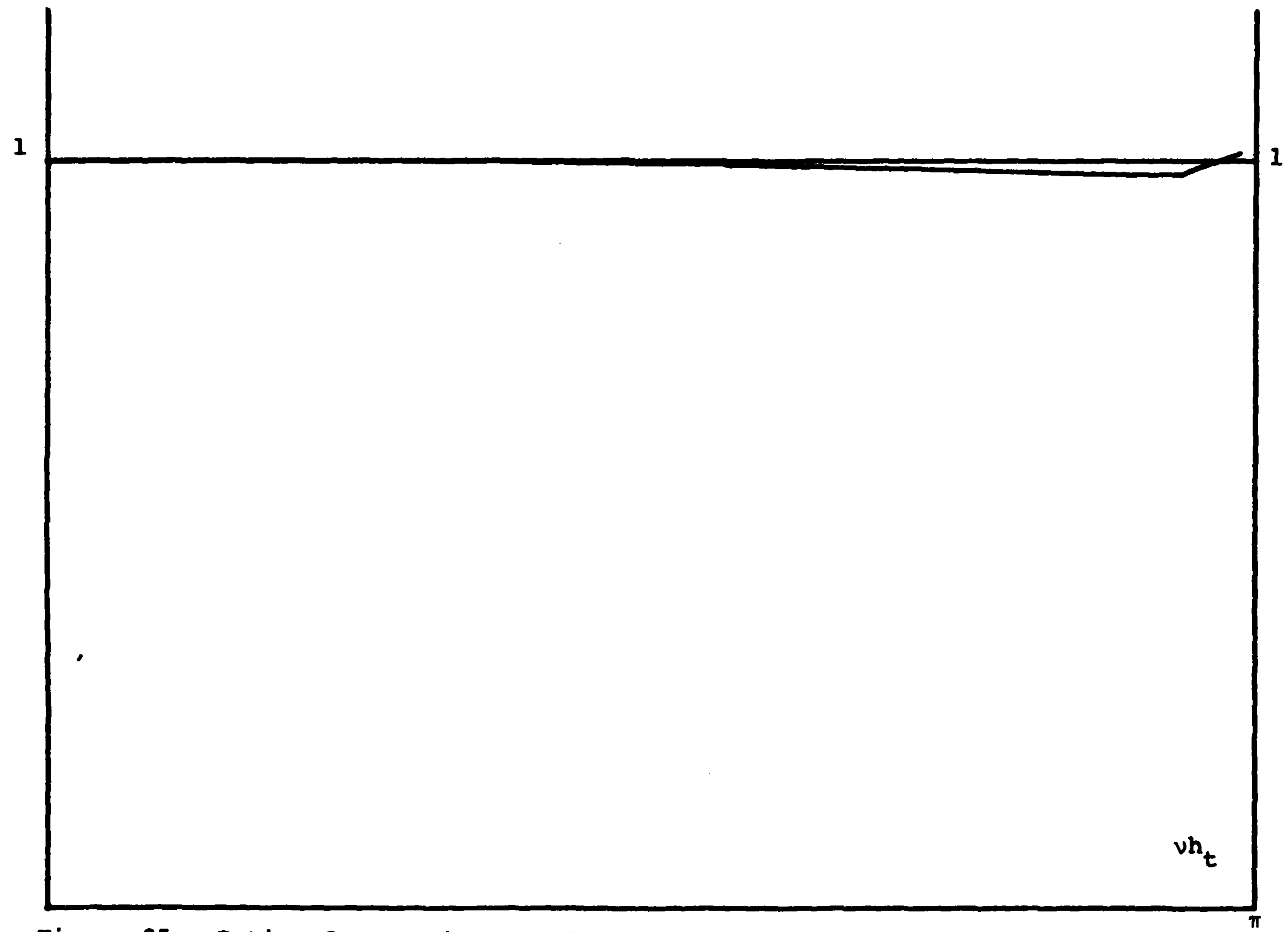


Figure 25. Ratio of Approximate and Exact Phase Velocity for the Time Dependence. Third Degree Shape Functions, Weight Functions Constant for $0 < \Delta t < h_t/2$ and $h_t/2 < \Delta t < h_t$.

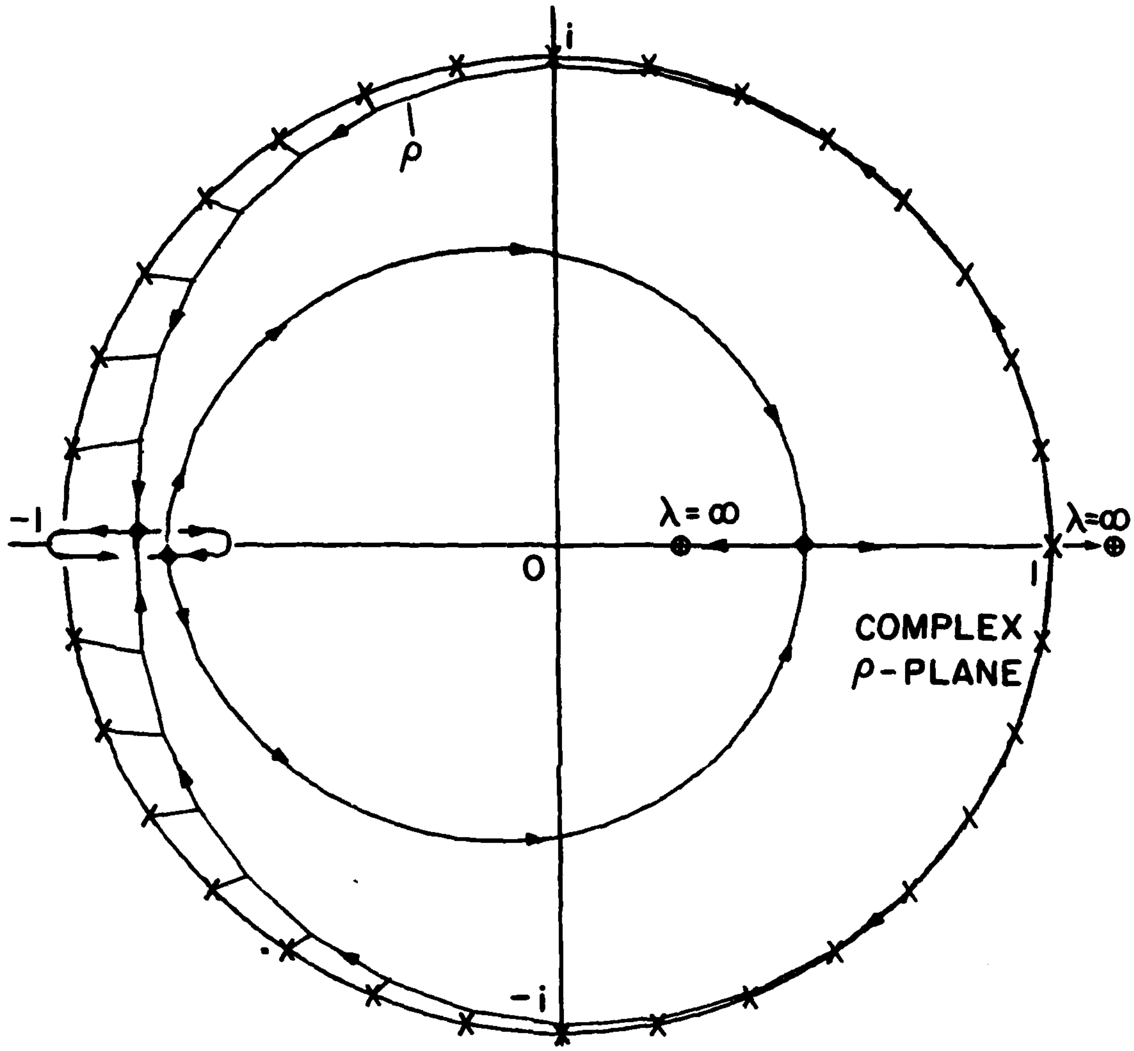


Figure 26. Amplification Factor ρ in the Complex Plane for Different Values of $\lambda^{1/2} = \nu h_t$. The arrows give the direction of increasing νh_t. In part of the figure the points ρ are connected with the ideal values $\exp(i\nu h_t)$. Third degree shape functions, weight functions constant for $0 < \Delta t < ch_t$ and $ch_t < \Delta t < h_t$, $c = 3/4$.

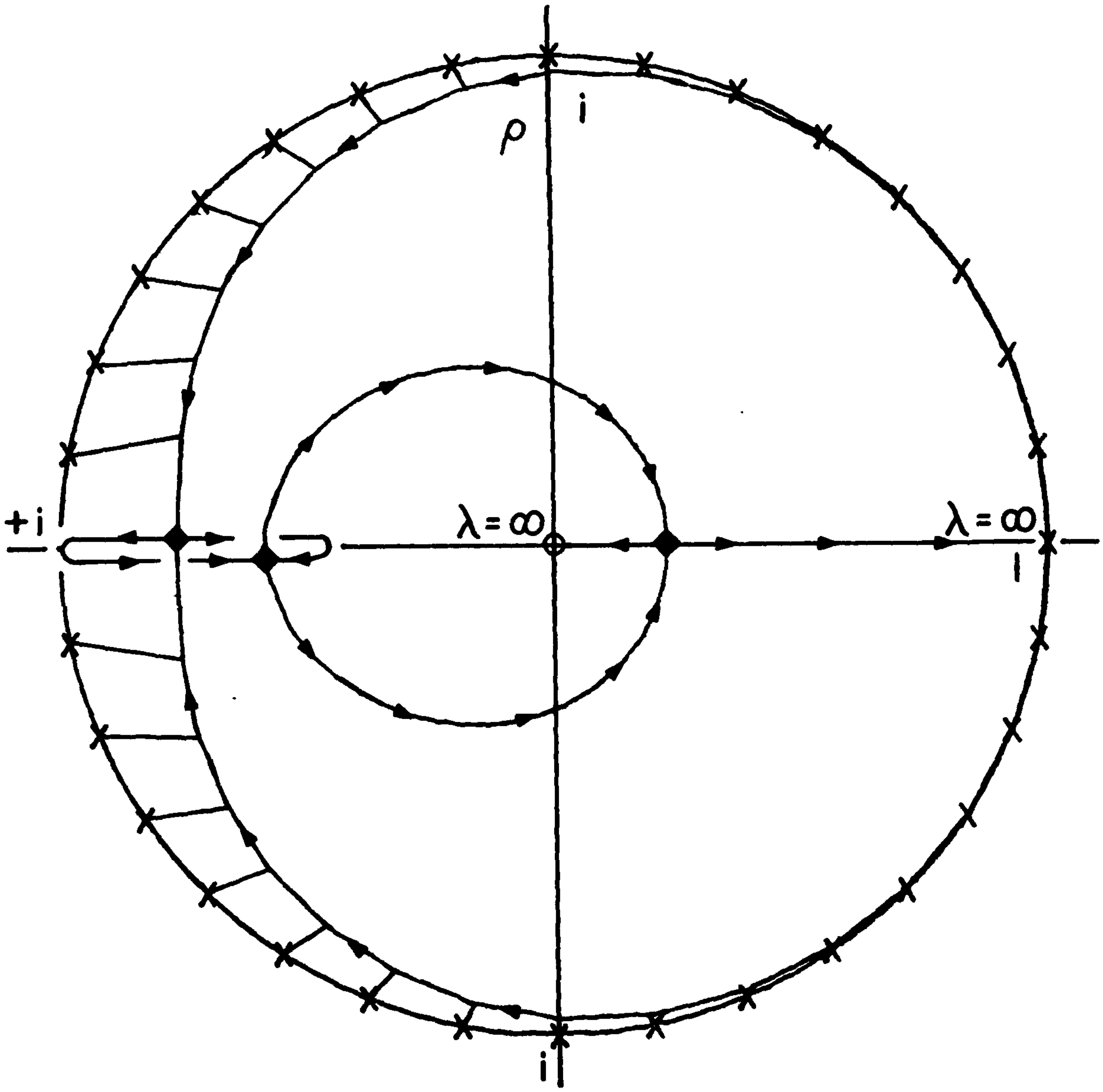


Figure 27. Amplification Factor ρ in the Complex Plane for Different Values of $\lambda^{1/2} = \nu h_t$. The arrows give the direction of increasing νh_t. In part of the figure the points ρ are connected with the ideal values $\exp(i\nu h_t)$. Third degree shape functions, weight functions constant for $0 < \Delta t < ch_t$ and $ch_t < \Delta t < h_t$, $c = 1$.

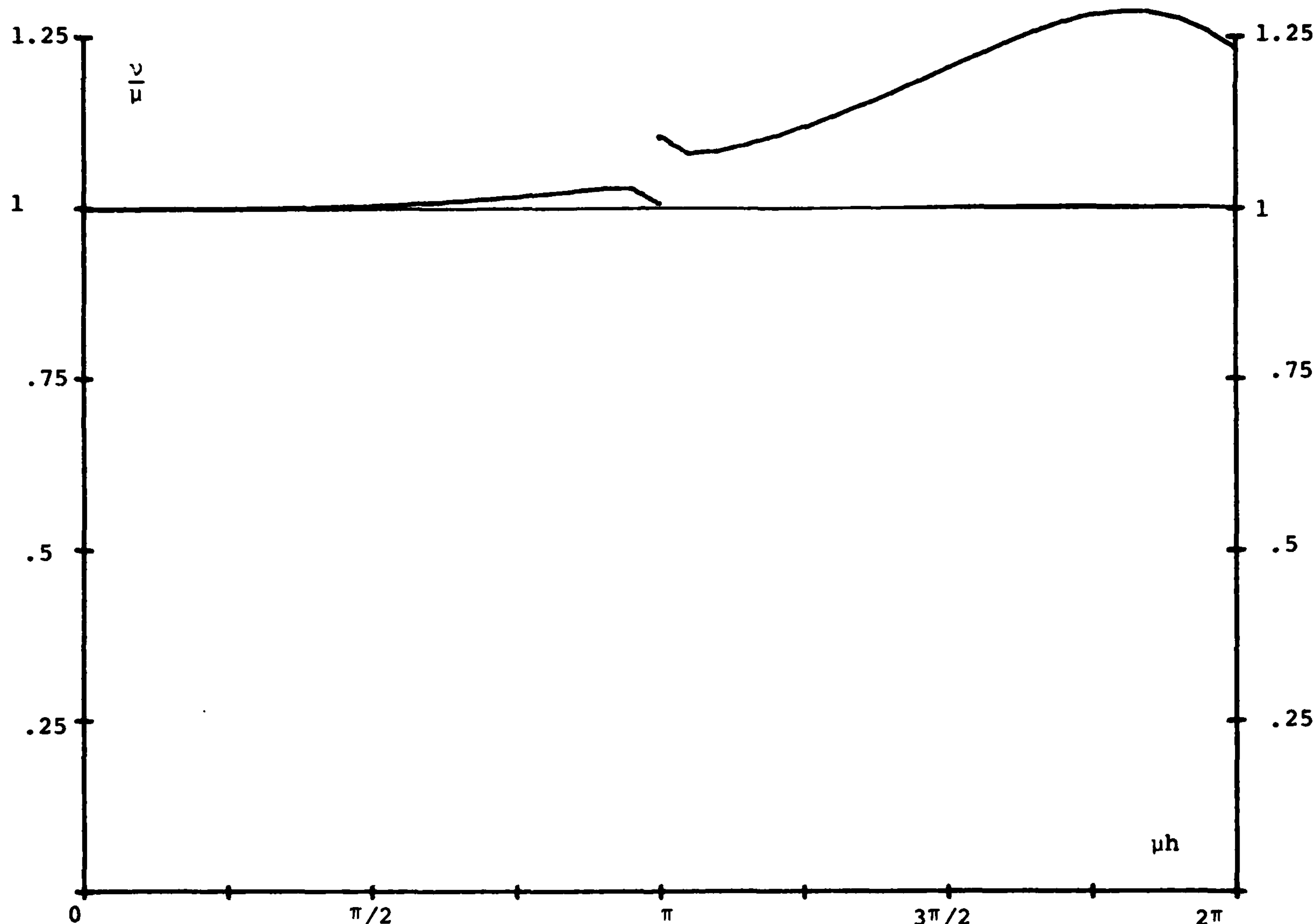


Figure 28. Ratio of Approximate and Exact Wave Velocities Third Degree Shape Functions. Weight Functions Give Exact Results for Very Long Waves.

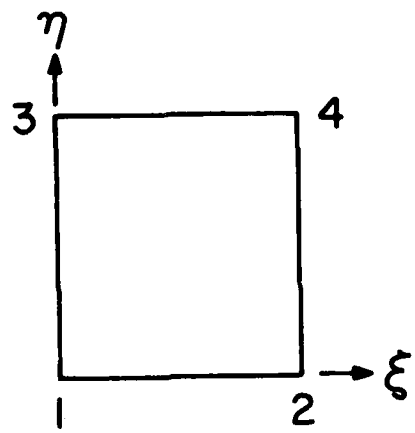


Figure 29. Characteristic Quadrangle.

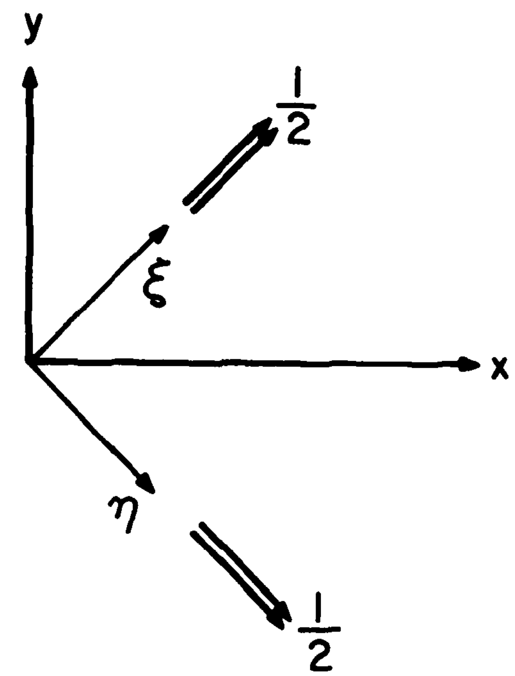


Figure 30. Characteristic Coordinates and Corrections to the Mass Flow Vector for Two-Dimensional Fundamental Solutions in Linearized flow with $M = \sqrt{2}$.

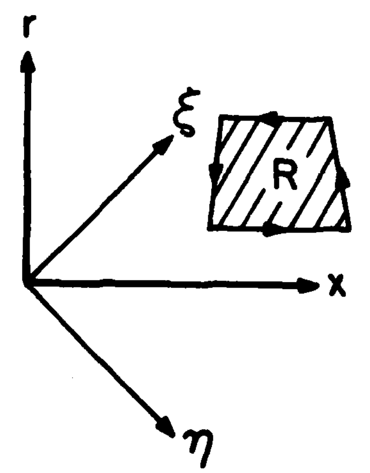


Figure 31. Characteristic Coordinates and region R in the ξ,η-plane for an Axisymmetric Linearized Flow with $M = \sqrt{2}$.

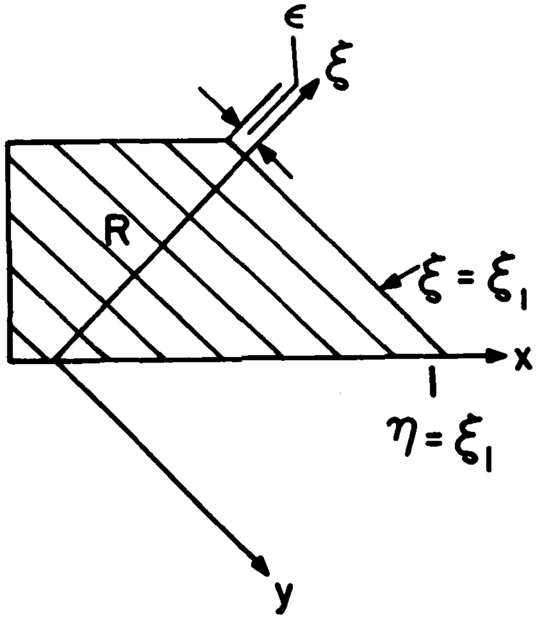


Figure 32. Region R for Which the Total Source Strength is Evaluated.

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